

Smooth Combs inside Hedgehogs

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Abstract. *We present a construction of a hedgehog containing smooth C^∞ combs. The hedgehog is a common hedgehog for a family of commuting non-linearisable holomorphic maps with a common indifferent fixed point. The comb is made up of smooth curves, and is transversally bi-Hölder regular.*

1. Introduction.

We consider the dynamics of a holomorphic diffeomorphism $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{R} - \mathbf{Q}$, defined in a neighbourhood of the indifferent irrational fixed point 0. The map f is said to be *linearisable* if there is a holomorphic change of variables $z = h(w) = w + O(w^2)$ such that

$$h^{-1} \circ f \circ h = R_\alpha$$

in a neighbourhood of the origin, where $R_\alpha(w) = e^{2\pi i\alpha}w$ is the rigid rotation.

The problem of *linearisation*, or determining when such an f is linearisable, has a long and interesting history, including the work of H. Cremer ([Cr1], [Cr2]) in the 1920's, C.L.Siegel ([Si]) in 1942, A.D.Bruno in the 1960's, and Yoccoz ([Y]), who resolved the problem of the optimal arithmetic condition on α for linearisability in 1987. In this article however we are primarily concerned with the structure of invariant sets for the dynamics near the fixed point, rather than the issue of linearisability.

When f is linearisable, f behaves like the rotation by angle $2\pi\alpha$ around 0 on the maximal domain of linearisation called the *Siegel disk* of f , which turns out to be a domain of the form $h(\{|w| < R\})$ where R is less than or equal to the radius of convergence of h around 0. In this case the compacts $h(\{|w| \leq r\})$ for $r < R$ are clearly completely invariant under f . However, even in the more general case where f is not necessarily linearisable, R.Perez-Marco found completely invariant continua for f which persist:

Theorem (Perez-Marco, [PM1]). *Let $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{R} - \mathbf{Q}$ be such that f and f^{-1} are defined and univalent on a neighbourhood of the closure of a Jordan domain $\Omega \subset \mathbf{C}$ containing 0. Then there exists a full compact connected set K contained in $\overline{\Omega}$ such that $0 \in K$, $K \cap \partial\Omega \neq \emptyset$ and $f(K) = f^{-1}(K) = K$.*

The invariant compacts K thus obtained are called *Siegel compacta*. A classical topological theorem of G.D.Birkhoff ([Bi]) does in fact guarantee, for planar homeomorphisms

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near Lyapunov unstable points, the existence of compacts, but which are either positive or negative invariant, not necessarily totally invariant. In the holomorphic setting one can thus improve to obtain totally invariant compacts. Moreover, though this will not be used in what follows, if the boundary of the Jordan domain Ω is C^1 smooth, then Perez-Marco has shown that such a compact K is in fact unique (see [PM5]). If K is not contained in the closure of a linearisation domain it is called a *hedgehog*. A *hedgehog* is called linearisable or non-linearisable depending on whether it contains a linearisation domain or not. We will be mostly concerned with non-linearizable hedgehogs in this paper. Perez-Marco has shown that these have no interior (see [PM5]). The structure of such hedgehogs is topologically complex; for example, Perez-Marco shows in [PM2] that they are not locally connected at any point different from the fixed point. Objects such as *combs* (homeomorphs of the product of a Cantor set and an interval) had been expected to be found within hedgehogs. Indeed this is stated by Perez-Marco in [PM4], where he also sketches how the techniques of "tube-log Riemann surfaces" used in [PM4] may be adapted to construct a hedgehog containing combs. We carry out this suggested construction in this article. The comb obtained is quite regular, being made up of smooth curves containing a dense set of analytic curves. Thus hedgehogs can exhibit some smoothness. We have the following:

Main Theorem. *There exists a Cantor set $C \subset \mathbf{R}$ and a family of commuting holomorphic maps $(f_t(z) = e^{2\pi it}z + O(z^2))_{t \in C}$, all defined and univalent on a common neighbourhood of the closed unit disk $\overline{\mathbf{D}}$, which have for irrational times t a common hedgehog $K \subset \overline{\mathbf{D}}$, such that K contains a comb \mathcal{E} , the homeomorphic image of $C_0 \times [1, 2]$ for an explicit Cantor set C_0 . The comb is made up of smooth C^∞ curves, and is bi-Hölder regular. More precisely, for the homeomorphism*

$$\begin{aligned} \Psi : C_0 \times [1, 2] &\rightarrow \mathcal{E} \\ (\theta, t) &\mapsto \Psi(\theta, t) \end{aligned}$$

we have:

- (a) For each fixed θ , the map $t \mapsto \Psi(\theta, t)$ is smooth.
- (b) Ψ is α -Hölder regular for all α , $0 < \alpha < 1$, and Ψ^{-1} is Lipschitz.

The maps f_t are non-linearisable for $t \in C \cap (\mathbf{R} - \mathbf{Q})$.

We note that since the hedgehog is non-linearisable, the comb obtained is non-trivial, in the sense that it is not contained inside the interior of the hedgehog. We first describe the construction, explaining how the hedgehog and comb are formed, and then carry out the proofs of the properties listed above.

For other examples of hedgehogs, we refer the reader to [PM3], where non-linearisable hedgehogs are constructed along with an explicit coding of the dynamics on the hedgehog, and [PM4], where linearisable hedgehogs are constructed.

2. The construction.

2.1 General idea of the construction.

The construction we will present is based on the techniques of "tube-log Riemann surfaces" invented by Perez-Marco in [PM6], where they were first used. They were used again in [PM3] and [PM4]. It is strongly advised to read this section in conjunction with [PM3] and [PM4]. As these techniques are explained in detail in [PM3] and especially in [PM4], and the basic elements of the construction are the same as in [PM4], to avoid repetition the description below is shorter and slightly informal.

We start with the linear flow of translations $(F_t(z) = z+t)_{t \in \mathbf{R}}$ on the upper half-plane \mathbf{H} (**note** : for convenience, instead of working with dynamics and dynamical objects such as Siegel disks, hedgehogs etc defined on the unit disk \mathbf{D} , we will actually work instead with their lifts to the upper half-plane via the universal covering $E : \mathbf{H} \rightarrow \mathbf{D} - 0, E(z) = e^{2\pi iz}$).

2.1.1) The Riemann surface \mathcal{S}_0 .

The idea is 'fold' the dynamics by introducing nonlinearities in the dynamics, such as fixed points; these result in a shrinking of the linearisation domain. This is accomplished by means of the following Riemann surface with distinguished charts \mathcal{S}_0 (see [PM4] section 1.2.1) :

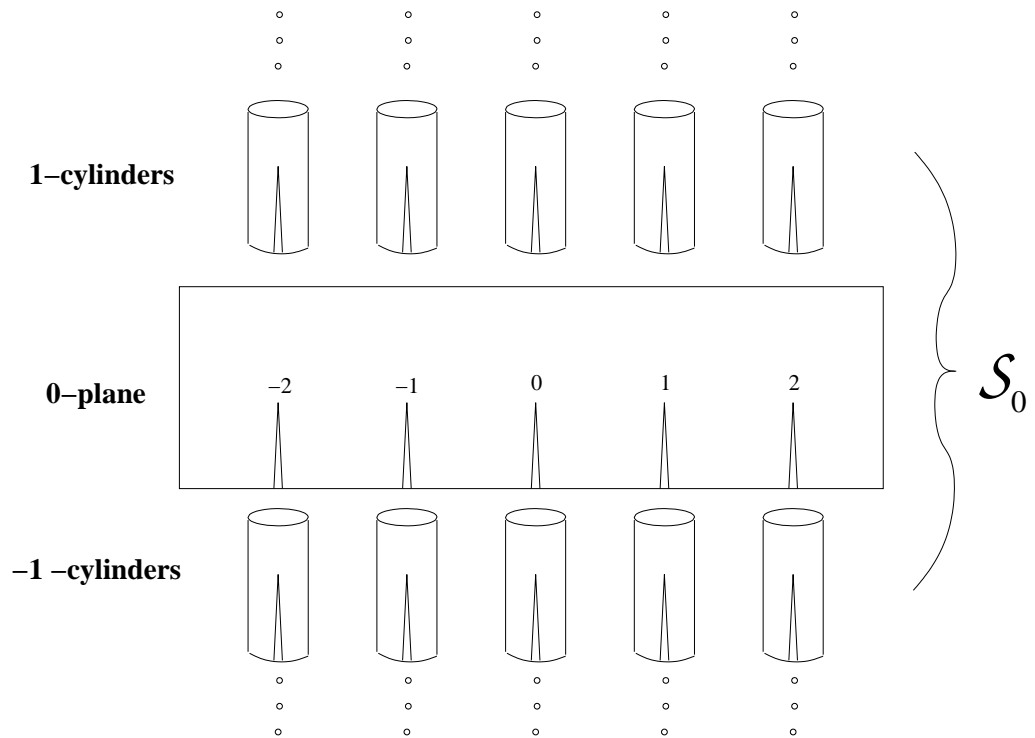


Figure 1

The surface \mathcal{S}_0 is formed from a copy of \mathbf{C} with 'slits' $[n, n - i\infty]$ at each of the integers $n \in \mathbf{Z}$, above and below each of which is pasted isometrically along the slits a family (indexed by \mathbf{Z}) of slit cylinders \mathbf{C}/\mathbf{Z} with slits $[0, 0 - i\infty]$. This construction gives not only a Riemann surface, but also a canonical set of charts for the surface, which allow us to write formulas for functions defined on the surface in terms of these canonical

coordinates. We note here that the surface \mathcal{S}_0 has 'ramification points of infinite order' at the points corresponding to \mathbf{Z} in the 0-plane and to 0 in the cylinders. These can be seen as points added in the completion of the flat metric on \mathcal{S}_0 (induced from the flat metrics on its building blocks). However it is important to note that these points do not belong to the surface \mathcal{S}_0 itself, nor can the complex structure on \mathcal{S}_0 be extended to these points.

2.1.2) Uniformization of the Riemann surface \mathcal{S}_0 .

The upper ends of the cylinders on the other hand do correspond to points for the complex structure. Adding in these points gives a simply connected Riemann surface which it is then not too hard to see is biholomorphic to \mathbf{C} . So \mathcal{S}_0 should be biholomorphic to the complex plane minus a doubly infinite discrete set of points; in fact, there is an explicit formula for the uniformization (see [PM4], section 1.2.3), which in terms of the distinguished charts on \mathcal{S}_0 is given by

$$K(z) = \frac{1}{2\pi i} \log(-\log(1 - e^{2\pi iz}))$$

The map K satisfies the properties

$$\begin{aligned} K(z+1) &= K(z) + 1 && \text{on all of the 0-sheet of } \mathcal{S}_0 \\ K(z) - z &\rightarrow 0 && \text{as } \text{Im } z \rightarrow +\infty \text{ in the 0-sheet of } \mathcal{S}_0 \end{aligned}$$

for the appropriate choices of the branches of the logarithms.

The images under K of horizontal lines just above the real line \mathbf{R} in the 0-plane are thus 1-periodic (with respect to parametrization by the x -coordinate) curves in \mathbf{C} ; since the ramification points at \mathbf{Z} in the 0-plane correspond to ends at infinity under the uniformization, these curves oscillate with amplitudes which tend to infinity as we take lines tending to the real line. The upper ends of the cylinders correspond under K to a discrete set of points $P \subset \mathbf{C}$; amongst these, those that correspond to the upper ends of the 1-cylinders lie below, but close to, the image of the real line. The figure below illustrates the image of the foliation of horizontal lines on \mathcal{S}_0 under K .

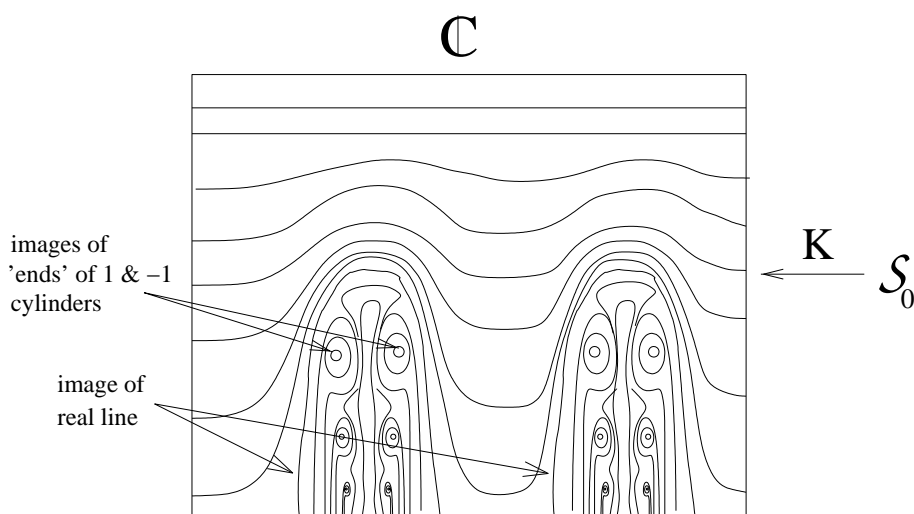


Figure 2

2.1.3) Lifting and folding of dynamics.

Consider the linear flow of translations $(F_t(z) = z + t)_{t \in \mathbf{R}}$ on the upper half-plane \mathbf{H} . The upper half-plane of the 0-plane of \mathcal{S}_0 projects univalently onto the upper half-plane \mathbf{H} , so we can lift each F_t to a map $\tilde{F}_t(z)$ from the upper half-plane of the 0-plane of \mathcal{S}_0 to itself, given in terms of the distinguished chart by the same formula $\tilde{F}_t(z) = z + t$. We would like to extend the lift analytically to a univalent map \tilde{F}_t of the surface to itself; however, it is not possible to extend to the points $\mathbf{Z} - t$ of the 0-plane or the points $0 - t$ of the cylinders, as these would have to be mapped to the ramification points, and moreover there is a non-trivial monodromy when continuing analytically around these points.

To overcome this problem, we can remove from the surface a small ϵ -neighbourhood (for the flat metric on \mathcal{S}_0) $\mathcal{I}(\epsilon) \subset \mathcal{S}_0$ of the ramification points. On $\mathcal{S}_0 - \mathcal{I}(\epsilon)$, for $|t| < \epsilon$, the aforementioned problem does not arise for the maps \tilde{F}_t , each of which then extends to a univalent map $\tilde{F}_t : \mathcal{S}_0 - \mathcal{I}(\epsilon) \rightarrow \mathcal{S}_0$, which is expressed in terms of the distinguished charts on \mathcal{S}_0 by the same formula $\tilde{F}_t(z) = z + t$ (it suffices in fact to remove only a set of horizontal slits $\mathbf{Z} + (\epsilon, \epsilon)$ near the ramification points, but for consistency with the rest of the article we consider removing neighbourhoods).

Let $T : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ be the automorphism given in the charts on \mathcal{S}_0 by $T(z) = z + 1$. We note that the lifts of the integer translations $z \mapsto z + n$ extend to all of \mathcal{S}_0 as the automorphisms $T^n : \mathcal{S}_0 \rightarrow \mathcal{S}_0$. These automorphisms commute with the maps \tilde{F}_t , hence we can also define for times $t' = n + t \in \mathbf{Z} + (-\epsilon, \epsilon)$ the lifts $\tilde{F}_{t'} : \mathcal{S}_0 - \mathcal{I}(\epsilon) \rightarrow \mathcal{S}_0$ by $\tilde{F}_{n+t} := T^n \circ \tilde{F}_t$. Hence we obtain a semi-flow (\tilde{F}_t) defined for times $t \in \mathbf{Z} + (-\epsilon, \epsilon)$ on a large part $\mathcal{S}_0 - \mathcal{I}(\epsilon)$ of the surface.

The 'upper ends' of the cylinders on the surface are points for the complex structure, which become fixed points (modulo the automorphism $T : \mathcal{S}_0 \rightarrow \mathcal{S}_0$) for the dynamics on the surface. Hence lifting the dynamics to the surface has the effect not only of extending its domain of definition, but also of introducing nonlinearities, such as fixed points.

To see this dynamics on the surface as dynamics on the plane, we use the uniformization $K : \mathcal{S}_0 \rightarrow \mathbf{C} - P$:

Conjugating the lifts \tilde{F}_t by K we obtain dynamics $K \circ \tilde{F}_t \circ K^{-1}$ on a large part $K(\mathcal{S}_0 - \mathcal{I}(\epsilon)) \subset \mathbf{C}$ of the plane. In particular any half-plane $\{\text{Im } z \geq -y\} \subset \mathbf{C}$ where $y \geq 0$ is contained in $K(\mathcal{S}_0 - \mathcal{I}(\epsilon))$ for ϵ small enough (see [PM4], section 1.2.4), so we can always ensure a large domain of definition for the dynamics. The dynamics flows along the foliation shown in Figure 2, and has as fixed points (modulo \mathbf{Z}) the points P corresponding under K to the upper ends of the cylinders. This dynamics is what we mean by the 'folded' dynamics.

2.1.4) Normalization of the uniformization.

The images under K of horizontal lines in the upper half-plane of the 0-sheet of \mathcal{S}_0 are graphs over the x -axis which are 1-periodic with respect to parametrization by the x -coordinate. For the image of a horizontal line at a height $\delta > 0$ in the 0-sheet, its peaks

and troughs occur at the points $K(n + 1/2 + i\delta)$ and $K(n + i\delta)$ respectively (where n runs over all integers). The amplitude of oscillation (the difference in height between peaks and troughs) is a strictly decreasing function of δ which tends to $+\infty$ as δ tends to 0, and tends to 0 as δ tends to $+\infty$.

We define a family of normalized uniformizations as follows:

First we place the ramification points of \mathcal{S}_0 slightly below the real axis instead of on the real axis; more precisely, for $\delta > 0$ we denote by \mathcal{S}_δ the Riemann surface obtained by the same construction as \mathcal{S}_0 , pasting a plane and cylinders isometrically along slits, but with ramification points placed at $\mathbf{Z} - i\delta$ in the 0-plane and $0 - i\delta$ in the cylinders. The uniformization of this surface is given by the map $z \mapsto K(z + i\delta)$ instead of $z \mapsto K(z)$. The image under this uniformization of the real line \mathbf{R} in the 0-sheet of \mathcal{S}_δ is a 1-periodic graph with an amplitude of oscillation that is large when δ is small. For small δ , the troughs of this curve have large negative imaginary parts and lie well below the real axis; in applications however, we would like the image of the real line to lie above the x -axis, with the troughs touching the x -axis, and so we define the uniformization $K_\delta : \mathcal{S}_\delta \rightarrow \mathbf{C}$ of \mathcal{S}_δ by

$$K_\delta(z) := K(z + i\delta) - i \times \text{"height of the troughs"}$$

where by the "height of the troughs" we mean

$$\text{"height of the troughs"} = \text{Im } K(n + i\delta) = -\frac{1}{2\pi} \log(-\log(1 - e^{-2\pi\delta}))$$

(where n here is any integer). The curve $K_\delta(\mathbf{R})$ then lies in the upper half-plane of \mathbf{C} with troughs lying on the real axis at the points $K_\delta(n + 0i) = n + 0i$, $n \in \mathbf{Z}$; the image of the upper half-plane of the 0-sheet of \mathcal{S}_δ under K_δ , which we denote by $K_\delta(\mathbf{H})$ is the region in the upper half-plane of \mathbf{C} bounded by $K_\delta(\mathbf{R})$.

In applications it will also be useful to have the image of the real line oscillate at higher frequencies, more precisely with period $1/a$ instead of period 1 for integers $a \geq 1$; to do this we consider the uniformizations $\frac{1}{a}K_\delta$ obtained by dividing K_δ by integers $a \geq 1$. The image of the real line, the curve $\frac{1}{a}K_\delta(\mathbf{R})$, is then a $1/a$ -periodic curve. Finally, in order to control the amplitude of oscillation, we note that given $a \geq 1$ integer and $h > 2$ real, there exists a unique $\delta = \delta(a, h)$ (as is shown in [PM4] section 1.2.5), with $0 < \delta < 1/2$, such that the amplitude of oscillation of $K_\delta(\mathbf{R})$ is $a \cdot h$, and hence that of $\frac{1}{a}K_\delta(\mathbf{R})$ is h . Moreover $\delta(a, h) \rightarrow 0$ for fixed h when $a \rightarrow \infty$. The constant δ satisfies (see [PM4] section 1.2.5)

$$\frac{\log(1 + e^{-2\pi\delta})}{-\log(1 - e^{-2\pi\delta})} = e^{-2\pi ah}$$

The peaks and troughs of the curve $\frac{1}{a}K_\delta(\mathbf{R})$ occur at the points $\frac{1}{a}K_\delta(n + 1/2 + 0i) = n + 1/2 + ih$ and $\frac{1}{a}K_\delta(n + 0i) = n + 0i$ respectively, and we have $\text{Max}_{x \in \mathbf{R}} \text{Im } \frac{1}{a}K_\delta(x) = \text{Im } \frac{1}{a}K_\delta(n + 1/2) = h$, $\text{Min}_{x \in \mathbf{R}} \text{Im } \frac{1}{a}K_\delta(x) = \text{Im } \frac{1}{a}K_\delta(n) = 0$, for any integer $n \in \mathbf{Z}$.

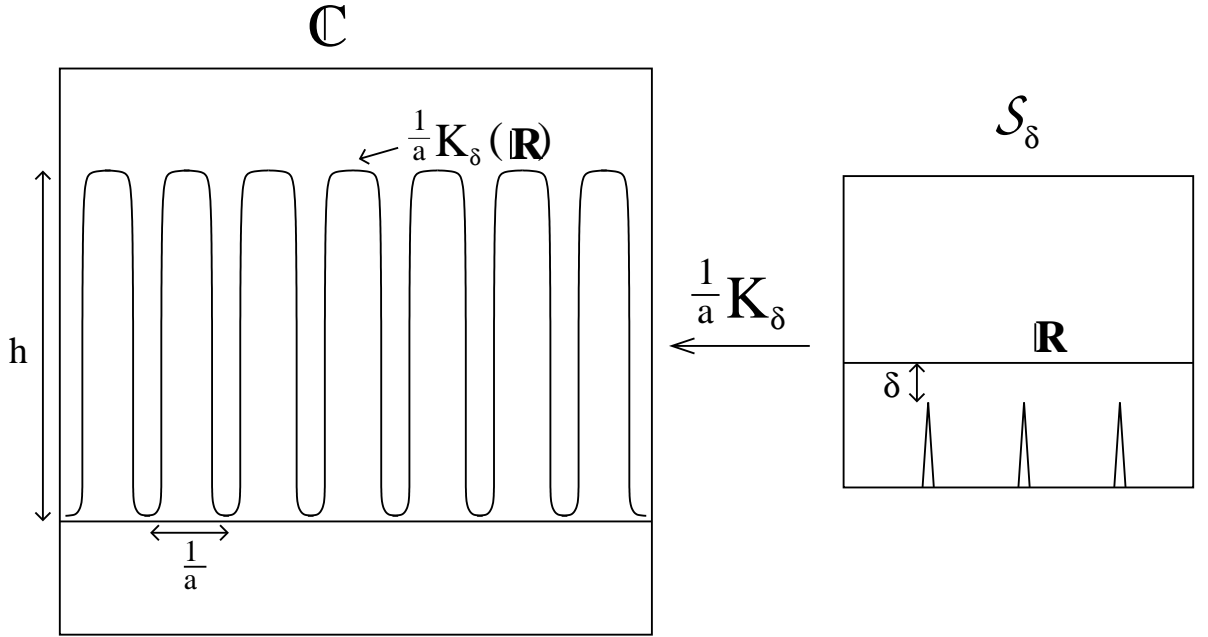


Figure 3.

We thus have a family of normalized uniformizations $\frac{1}{a}K_\delta$ depending on two parameters $a \geq 1$ an integer and $h > 2$ a real, where here δ is understood to be $\delta = \delta(a, h)$. The maps $\frac{1}{a}K_\delta$ have the following properties :

- (1) $\frac{1}{a}K_\delta$ is univalent on a neighbourhood of $\overline{\mathbf{H}}$, mapping \mathbf{H} into itself.
- (2) $\frac{1}{a}K_\delta(z + 1) = \frac{1}{a}K_\delta(z) + \frac{1}{a}$
- (3) $\frac{1}{a}K_\delta(i\mathbf{R}_{\geq 0}) = i\mathbf{R}_{\geq 0}$, $(\frac{1}{a}K_\delta)'(z) > 0$ for $z \in i\mathbf{R}_{\geq 0}$
- (4) There exists a constant C independent of a and $h > 2$ such that

$$\left| \frac{1}{a}K_\delta(z) - \left(\frac{1}{a}z + ih \right) \right| \leq \frac{C}{a}$$

for all z with $\text{Im } z > 1$.

(5) Let $z_0 \in \mathbf{C}$ denote the limit of $\frac{1}{a}K_\delta(z)$ when $z \rightarrow \infty$ in \mathcal{S}_δ through the upper end of one of the 1-cylinders. There exists a constant C_1 independent of a and $h > 2$ such that

$$\text{Im } z_0 \geq h - \frac{C_1}{a}$$

(note that the images under $\frac{1}{a}K_\delta$ of the upper ends of the 1-cylinders differ from each other by $1/a$ -translations, so $\text{Im } z_0$ is independent of the choice of 1-cylinder). We note that, for the dynamics obtained by conjugation by $\frac{1}{a}K_\delta$, these points are no longer necessarily fixed points (modulo \mathbf{Z}), but rather periodic points (modulo \mathbf{Z}) of period a .

We note that the properties (1)-(4) refer to $\frac{1}{a}K_\delta(z)$ for z in the 0-plane of \mathcal{S}_δ , while (5) refers to the 1-cylinders. These properties appear in [PM4], section 1.2.5, but can also

be checked by elementary computations . We note that condition **(5)** allows us to control the heights in the upper half-plane of the non-linearities of the dynamics obtained by conjugation by $\frac{1}{a}K_\delta$; indeed **(5)** says that the heights of the periodic orbits is approximately h , hence larger choices of h result in somehow smaller linearisation domains.

2.1.5) Construction of the limit dynamics.

The main idea is, roughly, to iterate the process described in section 2.1.3 above. We take a surface \mathcal{S}_{δ_1} , lift the linear flow of translations to \mathcal{S}_{δ_1} and conjugate by a uniformization $K_0 = \frac{1}{a_0}K_{\delta_1}$ to obtain folded dynamics as described in section 2.1.3. Having fixed \mathcal{S}_{δ_1} and K_0 , we take a surface \mathcal{S}_{δ_2} , lift the linear flow of translations to \mathcal{S}_{δ_2} and conjugate by a uniformization $K_1 = \frac{1}{a_1}K_{\delta_2}$; we then lift this folded dynamics to \mathcal{S}_{δ_1} and conjugate by the uniformization K_0 , to obtain a twice-folded dynamics, given by conjugating by the map K_0K_1 . We continue this way, choosing at each stage a surface \mathcal{S}_{δ_n} and a uniformization K_{n-1} , to obtain at stage n a dynamics given by conjugation by $K_0 \dots K_{n-1}$. At each stage the dynamics obtained is linearisable, but the linearisation domain decreases; the point is that, by choosing the appropriate uniformizations and controlling the geometry of the domains obtained at each stage, we should be able to obtain, in the limit, dynamics having an invariant domain with the desired geometry.

In the construction that follows, we define a sequence

$$K_n = \frac{1}{a_n}K_{\delta_{n+1}}, \quad \delta_{n+1} = \delta(a_n, h_n), \quad n \geq 0$$

of uniformizations depending on two sequences of parameters $(a_n)_{n \geq 0}$ and $(h_n)_{n \geq 0}$. The whole construction is completely determined by the sequences of parameters a_n and h_n , which we will indicate how to choose in the course of the construction. For the moment we assume that they are given, and construct the desired limit dynamics as follows:

Let $n \geq 1$.

We let $\mathcal{F}_{n,n} = (F_{n,n,t}(z) = z + t)_{t \in \mathbf{R}}$ be the linear flow of translations. As explained in section 2.1.3 above, we can lift the elements of this flow to univalent maps $\tilde{F}_{n,n,t}$ defined on the part of the surface \mathcal{S}_{δ_n} obtained by deleting an ϵ -neighbourhood $\mathcal{I}(\epsilon)$ of the ramification points and for times $t \in \mathbf{Z} + (-\epsilon, \epsilon)$. We take $\epsilon = 3a_n^{-1}$, and conjugate by the uniformization K_{n-1} of \mathcal{S}_{δ_n} to obtain a semi-flow $\mathcal{F}_{n-1,n} = (F_{n-1,n,t})_{t \in A_{n-1,n}}$ defined by

$$F_{n-1,n,a_{n-1}^{-1}t} := K_{n-1} \circ \tilde{F}_{n,n,t} \circ K_{n-1}^{-1}$$

where

$$a_{n-1}^{-1}t \in A_{n-1,n} := a_{n-1}^{-1}(\mathbf{Z} + [-2a_n^{-1}, 2a_n^{-1}]).$$

(we could in this step have taken a bigger set of times for $A_{n-1,n}$ such as $a_{n-1}^{-1}(\mathbf{Z} + (-3a_n^{-1}, 3a_n^{-1}))$, but in the steps that follow it will be necessary to restrict to smaller sets of times). We note that the time has been reparametrized in order to have the right behaviour at $+i\infty$, when $\text{Im } z \rightarrow +\infty$,

$$F_{n-1,n,t}(z) = z + t + o(1).$$

These maps are defined on $K_{n-1}(\mathcal{S}_{\delta_n} - \mathcal{I}(3a_n^{-1})) \subset \mathbf{C}$, which contains a large half-plane $\mathbf{H}_{-y} = \{ \operatorname{Im} z \geq -y \}$, where $y \rightarrow +\infty$ as $a_n \rightarrow +\infty$.

We would now like to lift the semi-flow $\mathcal{F}_{n-1,n}$ to the surface $\mathcal{S}_{\delta_{n-1}}$ and conjugate by the uniformization K_{n-2} to obtain a semi-flow $\mathcal{F}_{n-2,n}$. The difference with the previous case is that we are now starting with a nonlinear flow instead of the linear flow of translations. However, by giving up some space in the lower region of the half-plane of definition \mathbf{H}_{-y} , we can work with maps close to linear ones; more precisely we have (see [PM4] Lemma 1.1.6 and Proposition 2.2.6), for $t \in A_{n-1,n}$, if $\operatorname{Im} z \geq -y + \frac{1}{2\pi} \log a_{n-1} + C$, (where C is a universal constant) then

$$|F_{n-1,n,t}(z) - (z + t)| < \frac{1}{a_{n-1}}$$

Since $y \rightarrow +\infty$ as $a_n \rightarrow +\infty$, we observe that once a_{n-1} is fixed, by choosing a_n large enough we can ensure that this inequality holds in as large a half-plane $\mathbf{H}_{-\tilde{y}}$ as desired, where $-\tilde{y} = -y + \frac{1}{2\pi} \log a_{n-1} + C$.

Now consider the region $\mathcal{J}(-\tilde{y}) = \{z \in \mathcal{S}_{\delta_{n-1}} : \operatorname{Im} z \leq -\tilde{y}\} \subset \mathcal{S}_{\delta_{n-1}}$ (note that there is a well-defined function $\operatorname{Im} z$ on $\mathcal{S}_{\delta_{n-1}}$). Since the estimate above holds on the complement $\mathcal{S}_{\delta_{n-1}} - \mathcal{J}(-\tilde{y})$ of this region, it should be clear that if we now remove as well an ϵ -neighbourhood $\mathcal{I}(\epsilon)$ of the ramification points, this time with $\epsilon = 3a_{n-1}^{-1}$, then we can define, for $t \in \mathbf{Z} + A_{n-1,n} \cap [-2a_{n-1}^{-1}, 2a_{n-1}^{-1}]$, the lifts of elements of $\mathcal{F}_{n-1,n}$ as univalent maps $\tilde{F}_{n-1,n,t} : \mathcal{S}_{\delta_{n-1}} - (\mathcal{J}(-\tilde{y}) \cup \mathcal{I}(3a_{n-1}^{-1})) \rightarrow \mathcal{S}_{\delta_{n-1}}$.

We can now proceed as before and define the semi-flow $\mathcal{F}_{n-2,n} = (F_{n-2,n,t})_{t \in A_{n-2,n}}$ by

$$F_{n-2,n,a_{n-2}^{-1}t} := K_{n-2} \circ \tilde{F}_{n-1,n,t} \circ K_{n-2}^{-1}$$

where

$$a_{n-2}^{-1}t \in A_{n-2,n} := a_{n-2}^{-1}(\mathbf{Z} + A_{n-1,n} \cap [-2a_{n-1}^{-1}, 2a_{n-1}^{-1}])$$

(the times have again been reparametrized appropriately so that $F_{n-2,n,t}(z) = z + t + o(1)$ when $\operatorname{Im} z \rightarrow +\infty$).

These maps are defined on $K_{n-2}(\mathcal{S}_{\delta_{n-1}} - (\mathcal{J}(-\tilde{y}) \cup \mathcal{I}(3a_{n-1}^{-1}))) \subset \mathbf{C}$, which contains a large half-plane $\mathbf{H}_{-y'} = \{ \operatorname{Im} z \geq -y' \}$, where $y' \rightarrow +\infty$ as $a_{n-1} \rightarrow +\infty$ and $\tilde{y} \rightarrow \infty$ (see [PM4] Proposition 2.2.8).

Following this same procedure of lifting and conjugating the dynamics, we can define successively the semi-flows $\mathcal{F}_{n-3,n}, \mathcal{F}_{n-4,n}, \dots, \mathcal{F}_{0,n}$, with $\mathcal{F}_{j,n} = (F_{j,n,t})_{t \in A_{j,n}}$ defined for times $t \in A_{j,n}$ where $A_{j,n}$ is defined inductively by

$$A_{j,n} := a_j^{-1}(\mathbf{Z} + A_{j+1,n} \cap [-2a_{j+1}^{-1}, 2a_{j+1}^{-1}])$$

for $j = n-1, \dots, 0$ and $A_{n,n} = \mathbf{R}$. We note here that the sets $A_{0,n}$ form a decreasing sequence, ie $A_{0,n+1} \subset A_{0,n}$, $n \geq 1$; indeed,

$$A_{n-1,n+1} = a_{n-1}^{-1}(\mathbf{Z} + A_{n,n+1} \cap [-2a_n^{-1}, 2a_n^{-1}]) \subset a_{n-1}^{-1}(\mathbf{Z} + [-2a_n^{-1}, 2a_n^{-1}]) = A_{n-1,n}$$

from which it follows similarly that

$$\begin{aligned} A_{n-2,n+1} &\subset A_{n-2,n} \\ A_{n-3,n+1} &\subset A_{n-3,n} \\ &\dots \\ A_{0,n+1} &\subset A_{0,n} \end{aligned}$$

Thus we obtain a sequence of semi-flows $\mathcal{F}_{0,n} = (F_{0,n,t})_{t \in A_{0,n}}$, where each one is given by 'folding' the linear flow n times. Moreover, the a_n 's can be chosen inductively to grow fast enough so that all the semi-flows $\mathcal{F}_{0,n}$ are defined in the upper half-plane \mathbf{H} , or indeed in any fixed half-plane $\mathbf{H}_{-M} = \{ \operatorname{Im} z > -M \}$, as large as desired; indeed we can state as a proposition the following (which follows directly from [PM4] sections 1.3.1 and 2.2.3):

Proposition 2.1.1 *Given $M > 0$, there exists a sequence of conditions $(\mathcal{C}_n)_{n \geq 1}$ on the integers a_n of the form*

$$(\mathcal{C}_n) \quad a_n \geq C_n(a_0, \dots, a_{n-1}, h_0, \dots, h_{n-1}, M)$$

such that when they are fulfilled then all the semi-flows $(\mathcal{F}_{0,n})_{n \geq 1}$ are defined in the half-plane $\mathbf{H}_{-M} = \{ \operatorname{Im} z > -M \}$.

We observe thus a feature of the construction which is crucial when passing to the limit, namely that it allows us to successively paste nonlinearities to the dynamics without decreasing its domain of definition too much, allowing us to retain a half-plane \mathbf{H}_{-M} .

Assume that the a_n 's have been chosen so that all the semi-flows $\mathcal{F}_{0,n}$ are defined on \mathbf{H}_{-M} . From the sequence of semi-flows $\mathcal{F}_{0,n}$ we can extract a convergent subsequence (it always exists, see [PM4], section 1.1.2) converging normally for times $t \in A = \bigcap_{n \geq 0} A_{0,n}$ to get a limit semi-flow $\mathcal{F} = (F_t)_{t \in A}$ defined on the half-plane \mathbf{H}_{-M} for times $t \in A$. We note that elements of the limit semi-flow do indeed commute, since we can pass to the limit in the equation

$$F_{0,n_k,t} \circ F_{0,n_k,s}(z) = F_{0,n_k,s} \circ F_{0,n_k,t}(z)$$

(using uniform convergence in compact neighbourhoods of the points $z, F_t(z), F_s(z)$).

The Cantor set of times C in the main theorem is given by

$$C := \bigcap_{n \geq 0} A_{0,n} / \mathbf{Z}$$

If all the a_n 's are large enough, say for example $a_n \geq 5$ for all n , then the set C is indeed a Cantor set.

The semi-flow $(f_t)_{t \in C}$ of the main theorem is defined on a common neighbourhood $E(\mathbf{H}_{-M})$ of the closed unit disk $\overline{\mathbf{D}}$ via the equation

$$f_t \circ E = E \circ F_t, \quad t \in C$$

(where E is the universal covering $E : \mathbf{C} \rightarrow \mathbf{C} - \{0\}$, $E(z) = e^{2\pi iz}$).

As stated in section 2.1.4, each uniformization K_n maps the upper half-plane of the 0-plane of $\mathcal{S}_{\delta_{n+1}}$ into the upper half-plane \mathbf{H} . So we can consider its restriction to the upper-half plane of the 0-plane as a map $K_n : \mathbf{H} \rightarrow \mathbf{H}$ of \mathbf{H} into itself. If we consider the composition of these maps $K_0 \circ K_1 \circ \dots \circ K_n$, then it is clear that the set $K_0 K_1 \dots K_n(\overline{\mathbf{H}})$ is totally invariant under the semi-flow $\mathcal{F}_{0,n+1}$, and hence the set

$$\mathcal{H} = \bigcap_{n=0}^{\infty} K_0 K_1 \dots K_n(\overline{\mathbf{H}})$$

is then totally invariant under the limit semi-flow \mathcal{F} . Moreover, \mathcal{H} , being the decreasing limit of the domains $K_0 K_1 \dots K_n(\overline{\mathbf{H}})$, is full, and $\mathcal{H} \cap \partial\mathbf{H} \neq \emptyset$ (since $0 \in \mathcal{H}$). Thus \mathcal{H} is in fact a common hedgehog for irrational elements of the limit semi-flow \mathcal{F} . The hedgehog K of the semi-flow $(f_t)_{t \in \mathbf{C}}$ in the main theorem is given by

$$K = E(\mathcal{H})$$

It is important to note in this construction that the only requirement imposed on the a_n 's so that the limit dynamics is defined on the upper half-plane is that they grow fast enough, as expressed by Proposition 2.1.1 above. Subject to this restriction, the construction still offers enough flexibility to construct hedgehogs with quite different geometries by varying the choices of the parameters a_n and h_n .

In the following sections we show how to choose the uniformizations K_n appropriately (by proper choices of the parameters a_n, h_n), and control the geometries of the domains $K_0 K_1 \dots K_n(\overline{\mathbf{H}})$, so that the hedgehog \mathcal{H} contains a comb with the desired geometry.

2.2 Formation of the comb.

The purpose of this section is to describe informally the idea behind the construction of the comb within the hedgehog; precise statements and proofs are given in the following section, 2.3.

We define

$$\begin{aligned} \text{The strip } S &:= \{1 \leq \text{Im } z \leq 2\} \\ \text{The vertical half-lines } \xi_x &:= \{x + iy : y \geq 0\}, x \in \mathbf{R} \end{aligned}$$

In this section we prove only the following Proposition:

Proposition 2.2.1 *Let $m \in \mathbf{Z}$ and $N \geq 0$ be integers. Then the curve $K_0 \dots K_N(\xi_m)$ is contained in the hedgehog \mathcal{H} .*

Proof :

The properties **(2)** and **(3)** of the maps $\frac{1}{a}K_\delta$ given in section 2.1.4 imply that for any integer $m \in \mathbf{Z}$ and any $n \geq 0$ we have

$$K_n(\xi_m) = \xi_{m/a_n}.$$

Thus if we consider a curve $K_0 \dots K_N(\xi_m)$, then for all $n > N$ we can write

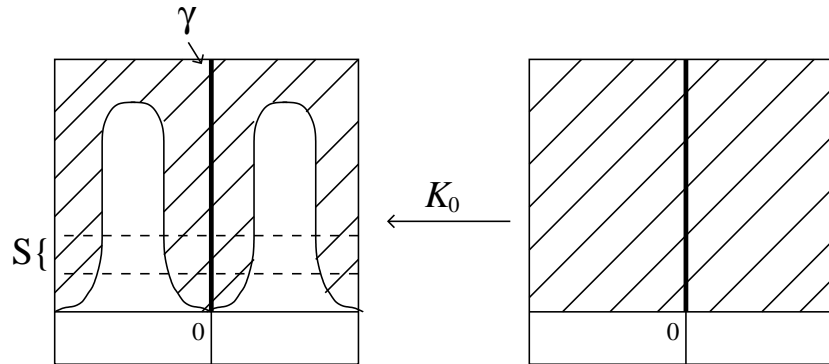
$$\begin{aligned} K_0 \dots K_N(\xi_m) &= K_0 \dots K_N \dots K_n(\xi_{m \cdot a_{N+1} \dots a_n}) \\ &\subset K_0 \dots K_N \dots K_n(\overline{\mathbf{H}}) \end{aligned}$$

and hence

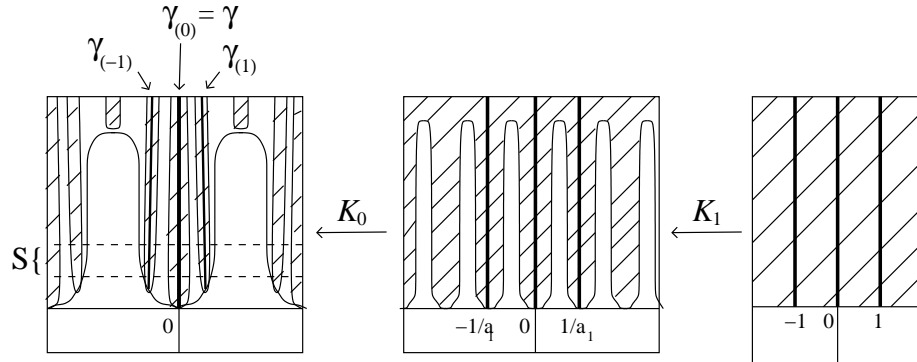
$$K_0 \dots K_N(\xi_m) \subset \bigcap_{n=N+1}^{\infty} K_0 \dots K_n(\overline{\mathbf{H}}) = \mathcal{H}$$

(since the domains in the intersection are decreasing, the intersection taken from $N+1$ to ∞ is the same as from 1 to ∞). \diamond

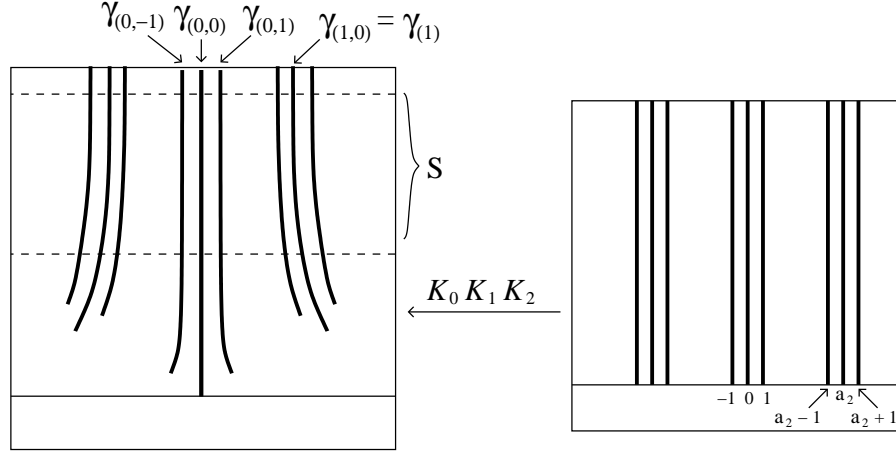
Thus all curves of the form $K_0 \dots K_N(\xi_m)$ are contained in the hedgehog \mathcal{H} . It is with such curves that we will construct the comb. We consider the formation of the comb in stages:



At stage $n = 0$, a_0 may be taken to be any positive integer greater than 5 and h_0 any real greater than 2. Only the map K_0 has been chosen, and we have only one curve $\gamma := K_0(\xi_0) = \xi_0$, as in the figure above. This gives the curve $\Gamma := \gamma \cap S$ within the strip S .



At the next stage $n = 1$, we need to choose the map K_1 . Now the images of the half-lines ξ_{-1}, ξ_1 under K_1 are the half-lines $\xi_{-1/a_1}, \xi_{1/a_1}$. So by choosing a_1 very large, since $1/a_1$ will be very small, the curves $K_0(K_1(\xi_{-1})) = K_0(\xi_{-1/a_1}), K_0(K_1(\xi_1)) = K_0(\xi_{1/a_1})$ will be "very close" to the curve $\gamma = K_0(\xi_0)$. More precisely, these curves will also pass through the strip S , be graphs over the imaginary axis, and if we fix an $r \geq 0$, then the parametrizations with respect to the imaginary coordinate will be close in C^r norm when $a_1 \rightarrow +\infty$. So at stage $n = 1$, we have three curves, $\gamma_{(-1)} := K_0(\xi_{-1/a_1}), \gamma_{(1)} := K_0(\xi_{1/a_1})$ and $\gamma_{(0)} := \gamma$ as in the figure above, which give the curves $\Gamma_{(-1)} := \gamma_{(-1)} \cap S, \Gamma_{(1)} := \gamma_{(1)} \cap S$ and $\Gamma_{(0)} := \gamma \cap S$ within the strip S .



The next stage $n = 2$ is similar. Under the map K_2 , for $\epsilon_0, \epsilon_1 \in \{-1, 0, 1\}$ the half-lines $\xi_{\epsilon_0 a_2 + \epsilon_1}$ are taken to the half-lines $\xi_{\epsilon_0 + \epsilon_1/a_2}$. Thus as in the previous stage, by choosing a_2 very large, since ϵ_1/a_2 will be very small, the curves $K_0(K_1(K_2(\xi_{\epsilon_0 a_2 + \epsilon_1}))) = K_0(K_1(\xi_{\epsilon_0 + \epsilon_1/a_2}))$ will be "very close" to the curves $K_0(K_1(\xi_{\epsilon_0}))$ from the previous stage. As before, all the curves will be graphs over the imaginary axis which pass through the strip S , with each new curve formed in this stage being close in C^{r+1} norm to a curve from the previous stage. At this stage $n = 2$ we get $9 = 3^2$ curves, $\gamma_{(\epsilon_0, \epsilon_1)} := K_0 K_1(\xi_{\epsilon_0 + \epsilon_1/a_2})$, such that $\gamma_{(\epsilon_0, 0)} = \gamma_{(\epsilon_0)}$, for $\epsilon_0, \epsilon_1 \in \{-1, 0, 1\}$. Each gives a curve $\Gamma_{(\epsilon_0, \epsilon_1)} := \gamma_{(\epsilon_0, \epsilon_1)} \cap S$ within the strip S .

Proceeding similarly, at each stage n , the map K_n is chosen, and we obtain 3^n curves indexed by n -tuples of the form $(\epsilon_0, \dots, \epsilon_{n-1}), \epsilon_j \in \{-1, 0, 1\}, j = 0, \dots, n-1$. Each curve is the image under $K_0 \dots K_{n-1}$ of a half-line ξ_x , for x of the form

$$x(\epsilon_0, \dots, \epsilon_{n-1}) := \epsilon_0 a_2 \dots a_{n-1} + \epsilon_1 a_3 \dots a_{n-1} + \dots + \epsilon_{n-2} + \frac{\epsilon_{n-1}}{a_n}$$

The curves are given by

$$\begin{aligned} \gamma_{(\epsilon_0, \dots, \epsilon_{n-1})} &:= K_0 K_1 \dots K_{n-1}(\xi_{x(\epsilon_0, \dots, \epsilon_{n-1})}) \quad \text{and satisfy} \\ \gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)} &= \gamma_{(\epsilon_0, \dots, \epsilon_{n-2})}, \quad \text{where } \epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\} \end{aligned}$$

As before, the curves are analytic, graphs over the imaginary axis and pass through the strip S . Thus each curve

$$\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})} := \gamma_{(\epsilon_0, \dots, \epsilon_{n-1})} \cap S$$

admits a (unique) real-analytic parametrization of the form

$$\begin{aligned}\sigma_{(\epsilon_0, \dots, \epsilon_{n-1})} : [1, 2] &\rightarrow \Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})} \\ t &\mapsto s_{(\epsilon_0, \dots, \epsilon_{n-1})}(t) + it\end{aligned}$$

This definition is made conditionally since the existence of these parametrizations depends on the choice of the parameters $(a_n), (h_n)$; that these parameters can indeed be chosen appropriately will be justified in the following section.

As explained above, choosing the a_n 's growing fast enough, each parametrization $\sigma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ will be very close to $\sigma_{(\epsilon_0, \dots, \epsilon_{n-2})}$. By making them very close in C^n norm say, we can ensure that for any (infinite) sequence $\tilde{\epsilon} = (\epsilon_0, \epsilon_1, \dots) \in \{-1, 0, 1\}^{\mathbf{N}}$, the sequence $(\sigma_{(\epsilon_0, \dots, \epsilon_n)})_{n \geq 0}$ is Cauchy in every C^r norm, and hence converges to a C^∞ parametrization, which we denote by $\sigma_{\tilde{\epsilon}}$, of a smooth curve $\Gamma_{\tilde{\epsilon}}$. We note that for finite sequences $\tilde{\epsilon} = (\epsilon_0, \dots, \epsilon_{n-1}, 0, 0, \dots)$, the curves $\Gamma_{\tilde{\epsilon}}$ coincide with the curves $\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ defined in the finite stages of the construction.

We define the set (which will be our comb)

$$\mathcal{C} := \bigcup_{\tilde{\epsilon} \in \{-1, 0, 1\}^{\mathbf{N}}} \Gamma_{\tilde{\epsilon}}$$

(this definition is also made conditionally since the existence of the curves $\Gamma_{\tilde{\epsilon}}$ for infinite sequences $\tilde{\epsilon}$ depends on the choice of the parameters $(a_n), (h_n)$); again, justification will be provided in the following section). The set \mathcal{E} which will be the comb of the main theorem is given by $\mathcal{E} = E(\mathcal{C})$ (where $E(z) = e^{2\pi iz}$). That these sets are indeed combs and contained in the hedgehog \mathcal{H} will be shown in section 3.

We note that points of \mathcal{C} can be represented naturally by pairs $(\tilde{\epsilon}, t)$, thus exhibiting for \mathcal{C} a product structure of the form Cantor set \times interval, with $\tilde{\epsilon}$ belonging to the abstract Cantor set $\{-1, 0, 1\}^{\mathbf{N}}$ with the product topology, and t to the interval $[1, 2]$.

To study the regularity of the comb \mathcal{C} however, we will parametrize it by the metric comb $C_0 \times [1, 2]$, where the Cantor set $C_0 \subseteq \mathbf{R}$ is given by embedding $\{-1, 0, 1\}^{\mathbf{N}}$ into \mathbf{R} , via the map

$$\begin{aligned}\Theta : \{-1, 0, 1\}^{\mathbf{N}} &\rightarrow \mathbf{R} \\ \tilde{\epsilon} &\mapsto \theta(\tilde{\epsilon}) := \frac{\epsilon_0}{a_0 a_1} + \frac{\epsilon_1}{a_0 a_1 a_2} + \dots + \frac{\epsilon_n}{a_0 \dots a_{n+1}} + \dots\end{aligned}$$

(the map is a homeomorphism onto its image C_0 if the a_n 's are large enough, for example if $a_n \geq 4$ for all n). For sequences $\tilde{\epsilon} = (\epsilon_0, \dots, \epsilon_n, 0, 0, 0, \dots)$ with finitely many non-zero terms we will denote $\theta(\tilde{\epsilon})$ by $\theta(\epsilon_0, \dots, \epsilon_n)$.

The homeomorphism $\Phi : C_0 \times [1, 2] \rightarrow \mathcal{C}$ is then given by

$$\begin{aligned}\Phi : C_0 \times [1, 2] &\rightarrow \mathcal{C} \\ (\theta(\tilde{\epsilon}), t) &\mapsto \sigma_{\tilde{\epsilon}}(t)\end{aligned}$$

(the homeomorphism Ψ in the main theorem is given by composing with the exponential, $\Psi = E \circ \Phi$). We remark again that this definition is conditional, depending on the choice of parameters a_n, h_n ; in the following section 2.3, we show how to choose the parameters so that this map Φ is well-defined, a homeomorphism, and satisfies the Hölder and Lipschitz conditions stated in the main theorem, which will be proved in section 3. We note also that, ensuring that the comb is strictly contained in a period strip for the exponential, such as $\{ -1/2 < \operatorname{Re} z < 1/2 \}$ (this will be achieved by proper choice of the parameters), it is not hard to check that these properties for Φ imply the same for Ψ .

2.3 Inductive choice of the parameters.

We let

$$G_n = u_n + iv_n := K_0 \dots K_n, \quad n \geq 0$$

We fix an increasing sequence $\alpha_n \uparrow 1$ (we will ensure at each stage n in the formation of the comb that Φ satisfies an α_n -Hölder estimate, so that in the limit Φ will be α -Hölder for all α). Fix a constant $M = 1$ (any $M > 0$ will do); the parameters will be chosen so that the limit semi-flow \mathcal{F} is defined on the half-plane \mathbf{H}_{-M} .

The following lemma will be used in the construction to ensure that Φ is injective and Φ^{-1} Lipschitz:

Lemma 2.3.1. *Given fixed constants $v, h > 0$ such that $0 < v < h$, for $a \geq 1$ and $x_0 \in \mathbf{Z}$, let $z_0 = x_0 + iy_0$, with y_0 depending on a , be such that $\operatorname{Im} \frac{1}{a} K_{\delta(a,h)}(z_0) = v$ (note that y_0 doesn't depend on x_0). Then for $z = x_0 + iy$ such that $0 \leq y \leq y_0$,*

$$\left| a \cdot \frac{1}{a} K'_{\delta(a,h)}(z) \right| = |K'_{\delta(a,h)}(z)| \rightarrow \infty \quad \text{as } a \rightarrow \infty.$$

uniformly in x_0, y .

Proof.

We first note that

$$\begin{aligned} K'(z) &= \frac{e^{2\pi iz}}{-\log(1 - e^{2\pi iz})} \cdot \frac{1}{(1 - e^{2\pi iz})} \\ &= \frac{1 - e^{-e^{2\pi iK(z)}}}{e^{2\pi iK(z)} \cdot e^{-e^{2\pi iK(z)}}} \\ &= \frac{e^{e^{2\pi iK(z)}} - 1}{e^{2\pi iK(z)}} \end{aligned}$$

Now

$$\frac{1}{a} K_{\delta}(z_0) = \frac{1}{a} (K(z_0 + i\delta) + iC_{\delta})$$

where $C_{\delta} \in \mathbf{R}$ is a constant depending on δ

$$= \frac{1}{a} (x_0 + K(i(y_0 + \delta)) + iC_{\delta})$$

so considering the imaginary part of the above expression gives

$$\frac{1}{a} (K(i(y_0 + \delta)) + iC_\delta) = iv,$$

so

$$\begin{aligned} K(i(y_0 + \delta)) &= i(av - C_\delta) \\ &= i(av - \frac{1}{2\pi} \log(-\log(1 - e^{-2\pi\delta}))) \\ &= i(av - \frac{1}{2\pi} \log(e^{2\pi ah} \log(1 + e^{-2\pi\delta}))) \\ &= i(av - ah - \frac{1}{2\pi} \log \log(1 + e^{-2\pi\delta})) \\ &= i(a(v - h) + O(1)) \text{ as } a \rightarrow \infty \end{aligned}$$

(we have used above the condition on δ given in section 2.1.4). Thus, letting $w_0 = e^{2\pi i K(i(y_0 + \delta))}$, we have

$$w_0 = e^{2\pi(a(h-v)+O(1))} \in \mathbf{R}_+ \text{ and } w_0 \rightarrow +\infty \text{ as } a \rightarrow \infty$$

so

$$\begin{aligned} \left| a \cdot \frac{1}{a} K'_{\delta(a,h)}(z_0) \right| &= |K'_{\delta(a,h)}(z_0)| \\ &= |K'(z_0 + i\delta)| \text{ (since } K_\delta(z) = K(z + i\delta) + \text{constant)} \\ &= |K'(i(y_0 + \delta))| \text{ (since } K(x_0 + z) = K(z) + x_0) \\ &= \frac{e^{w_0} - 1}{w_0} \\ &\rightarrow +\infty \text{ as } a \rightarrow \infty \end{aligned}$$

For $z = x_0 + iy$, $0 \leq y \leq y_0$, we note that $\text{Im } K(i(y + \delta)) \leq \text{Im } K(i(y_0 + \delta))$, so $w = e^{2\pi i K(i(y + \delta))} \geq w_0$, and

$$\left| a \cdot \frac{1}{a} K'_{\delta(a,h)}(z_0) \right| = \frac{e^w - 1}{w} \geq \frac{e^{w_0} - 1}{w_0},$$

a lower bound independent of x_0, y , hence the convergence is uniform in x_0, y . \diamond .

We can now begin the construction. At stage

n = 0 : Let $h_0 = 10$. We choose $a_0 \geq 5$ to satisfy the following:

Applying Lemma 2.3.1 above to $K_0 = \frac{1}{a_0} K_{\delta(a_0, h_0)}$, with $v = h_0 - 1/2 < h = h_0$ and $a = a_0$, we require that a_0 is chosen large enough so that

$$|a_0 \cdot K'_0(z)| > 1 \text{ for all } z = x + iy, x \in \mathbf{Z}, 0 \leq y \leq Y_1$$

where Y_1 depending on a_0 is chosen such that $\text{Im } K_0(iY_1) = h_0 - 1/2$.

The parameters a_0 and h_0 having been chosen, the map K_0 is now fixed. We need to choose additional parameters so that the induction can continue:

Using property **(4)** of the maps $\frac{1}{a}K_\delta$, we choose $h_1 > 2$ such that $\text{Im } K_0(x+i(h_1-1)) > h_0 + 1$ for all $x \in \mathbf{R}$. We choose $h'_1 > 0$ such that $2 < \text{Im } K_0(0 + ih'_1)$. Also, we have $0 = \text{Im } K_0(0 + i0) < 1$; so by continuity, we can choose $\tau_1 > 0$ such that

$$2 < \text{Im } K_0(x + ih'_1), \text{Im } K_0(x + i0) < 1 \quad \text{for } |x| \leq \tau_1$$

By property **(3)** of the maps $\frac{1}{a}K_\delta$, $\frac{\partial v_0}{\partial y} > 0$ on the vertical segment $[0 + i0, 0 + ih'_1]$; thus by continuity and compactness of the segment, we also take τ_1 small enough to have

$$\frac{\partial v_0}{\partial y}(x + iy) > 0 \quad \text{for } |x| \leq \tau_1, 0 \leq y \leq h'_1$$

The reason for imposing the conditions above is to guarantee that for $|x| \leq \tau_1$ the curves $K_0(\xi_x)$ pass through the strip S and, in S , are graphs over the imaginary axis, so that in the next stage, we can choose a_1 such that $1/a_1 < \tau_1$ to ensure the same is true for the curves $\gamma_{(-1)} = K_0(\xi_{-1/a_1})$, $\gamma_{(1)} = K_0(\xi_{1/a_1})$.

By continuity and compactness of the segment $[0 + i0, 0 + iY_1]$, we also take τ_1 small enough so that

$$|a_0 \cdot K'_0(z)| > 1 \quad \text{for all } z = x + iy, |x| \leq \tau_1, 0 \leq y \leq Y_1$$

The set $K_0(\{z = x + iy : |x| \leq \tau_1, 0 \leq y \leq Y_1\})$ is a neighbourhood of the compact segment $[0 + 1i, 0 + 2i]$ (which is the first curve $\Gamma = \gamma \cap S$ occurring in the construction as described in section 2.2). So we can choose an open rectangle R such that

$$[0 + 1i, 0 + 2i] = \Gamma \subset R \subset K_0(\{z = x + iy : |x| \leq \tau_1, 0 \leq y \leq Y_1\})$$

We define the domains D_1 and U by

$$U := K_0^{-1}(R), \quad D_1 = U.$$

We observe that $\Gamma \subset K_0(U)$, $K_0(U) = R$, and $U \subset \{z = x + iy : |x| \leq \tau_1, 0 \leq y \leq Y_1\}$ so that $|K'_0(z)| > 1/a_0$ in U .

n ≥ 1 : We assume the parameters $a_0, \dots, a_{n-1}, h_0, \dots, h_{n-1}$ have been chosen, so the maps K_0, \dots, K_{n-1} are fixed. As induction hypothesis we also assume that:

1. $h_n > 2$ has been chosen such that $\text{Im } K_0 \dots K_{n-1}(x + i(h_n - 1)) = v_{n-1}(x + i(h_n - 1)) > h_0 + n$ for all $x \in \mathbf{R}$.

(This condition will allow us to ensure that all elements of the semi-flow $\mathcal{F}_{0,n}$ have a common periodic orbit at a height at least $h_0 + n$, so that elements of the limit flow will have periodic orbits above any given height and hence be nonlinearisable for irrational rotation numbers).

2. Constants $h'_n > 0$ and $\tau_n > 0$ have been chosen such that

$$\begin{aligned} v_{n-1}(x + i0) < 1, v_{n-1}(x + ih'_n) > 2 \quad \text{and} \\ \frac{\partial v_{n-1}}{\partial y}(x + iy) > 0 \\ \text{for } |x - x(\epsilon_0, \dots, \epsilon_{n-2}, 0)| \leq \tau_n, \quad 0 \leq y \leq h'_n, \\ \text{and } \epsilon_0, \dots, \epsilon_{n-2} \in \{-1, 0, 1\} \text{ (for } n = 1, \text{ we only have } |x| \leq \tau_1) \end{aligned}$$

(This condition is to ensure that the images $K_0 \dots K_{n-1}(\xi_x)$ of lines ξ_x lying in the rectangular strips $\{ |x - x(\epsilon_0, \dots, \epsilon_{n-2}, 0)| \leq \tau_n, \quad 0 \leq y \leq h'_n \}$ are 'vertical' enough, ie are graphs over the imaginary axis passing through S).

3. For all $\epsilon_0, \dots, \epsilon_{n-2} \in \{-1, 0, 1\}$ we have

$$|a_{n-1} \cdot K'_{n-1}(z)| > 1 \quad \text{for all } z = x + iy, \quad |x - x(\epsilon_0, \dots, \epsilon_{n-2}, 0)| \leq \tau_n, \quad 0 \leq y \leq Y_n$$

where Y_n is a constant which has been chosen such that $\text{Im } K_{n-1}(Y_n) = h_{n-1} - 1/2$ (for $n=1$, the above inequality holds for $|x| \leq \tau_1, \quad 0 \leq y \leq Y_1$).

4. Open sets $D_k, \quad k = 1, \dots, n$ have been chosen, of the form

$$D_k = \bigcup_{\epsilon_0, \dots, \epsilon_{k-2} \in \{-1, 0, 1\}} U(\epsilon_0, \dots, \epsilon_{k-2}), \quad 2 \leq k \leq n$$

(for $k = 1$ we only have $D_1 = U$) where each $U(\epsilon_0, \dots, \epsilon_{k-2})$ is a domain such that

$$\Gamma_{(\epsilon_0, \dots, \epsilon_{k-2})} \subset K_0 \dots K_{k-1}(U(\epsilon_0, \dots, \epsilon_{k-2}))$$

(for $k = 1$ we have $\Gamma \subset K_0(U)$), the image of $U(\epsilon_0, \dots, \epsilon_{k-2})$ under K_{k-1} is an open rectangle $R(\epsilon_0, \dots, \epsilon_{k-2})$,

$$K_{k-1}(U(\epsilon_0, \dots, \epsilon_{k-2})) = R(\epsilon_0, \dots, \epsilon_{k-2})$$

(for $k = 1$ we have $K_0(U) = R$) and

$$U(\epsilon_0, \dots, \epsilon_{k-2}) \subset \{ z = x + iy, \quad |x - x(\epsilon_0, \dots, \epsilon_{k-2})| \leq \tau_k, \quad 0 \leq y \leq Y_k \}$$

(for $k = 1$ we have $U \subset \{ z = x + iy : |x| \leq \tau_1, \quad 0 \leq y \leq Y_1 \}$) so that

$$|K'_{k-1}(z)| > 1/a_{k-1}, \quad z \in U(\epsilon_0, \dots, \epsilon_{k-2})$$

(for $k = 1$ we have $|K'_0(z)| > 1/a_0$ in U). Finally, we also assume a nesting condition,

$$K_{k-1}(U(\epsilon_0, \dots, \epsilon_{k-2})) = R(\epsilon_0, \dots, \epsilon_{k-2}) \subset U(\epsilon_0, \dots, \epsilon_{k-3})$$

(for $k = 1$ this condition is empty, while for $k = 2$ we request $K_1(U(\epsilon_0)) = R(\epsilon_0) \subset U$).

These hypotheses imply the following proposition, which will be used later in proving that Φ is injective and Φ^{-1} is Lipschitz:

Proposition 2.3.2. *For $n \geq 1$ and $z, z' \in U(\epsilon_0, \dots, \epsilon_{n-2})$ ($z, z' \in U$ for $n = 1$), we have*

$$|K_{n-1}(z) - K_{n-1}(z')| \geq \frac{|z - z'|}{a_{n-1}}$$

Lemma 2.3.3. *Let $f : U_1 \rightarrow U_2$ be a holomorphic diffeomorphism between two open sets $U_1, U_2 \subset \mathbf{C}$ such that $|f'(z)| \geq C > 0$ for all $z \in U_1$, for some constant $C > 0$. Suppose U_2 is convex. Then*

$$|f(z) - f(z')| \geq C|z - z'|$$

for all $z, z' \in U_1$.

Proof :

Since U_2 is convex, given $z, z' \in U_1$ the straight line segment joining $f(z)$ to $f(z')$ lies in U_2 , thus applying the Mean value Theorem to f^{-1} gives the desired inequality. \diamond

Proof of Proposition 2.3.2 :

The result follows from applying the above lemma to the map $K_{n-1} : U(\epsilon_0, \dots, \epsilon_{n-2}) \rightarrow R(\epsilon_0, \dots, \epsilon_{n-2})$, observing that by the induction hypothesis **4.**,

$$|K'_{n-1}(z)| > 1/a_{n-1}, \quad z \in U(\epsilon_0, \dots, \epsilon_{n-2}),$$

and that $R(\epsilon_0, \dots, \epsilon_{n-2})$, being a rectangle, is convex. \diamond

We now outline the conditions required on a_n :

(a) We require that $a_n \geq 5$ is chosen large enough to satisfy the condition (C_n) ,

$$a_n \geq C_n(a_0, \dots, a_{n-1}, h_0, \dots, h_{n-1}, M)$$

(note $a_n \geq 5$ ensures that C and C_0 are Cantor sets).

(b) For the existence of the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-1})} : [1, 2] \rightarrow \Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ of the curves formed at stage n :

We require that $1/a_n < \tau_n$. The induction hypothesis **2.** on τ_n and h'_n given above ensures that then the curves $\gamma_{(\epsilon_0, \dots, \epsilon_n)}$ will pass through S , and, in S , be graphs over the imaginary axis. Thus the existence of the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-1})} : [1, 2] \rightarrow \Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ such that $\text{Im } \sigma_{(\epsilon_0, \dots, \epsilon_{n-1})}(t) = t$ is guaranteed.

(c) For the curves formed at stage n to be C^n close to those of stage $n - 1$:

Since $x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) = x(\epsilon_0, \dots, \epsilon_{n-2}, 0) + \epsilon_{n-1}/a_n \rightarrow x(\epsilon_0, \dots, \epsilon_{n-2}, 0)$ as $a_n \rightarrow \infty$, clearly the parametrizations

$$\phi_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})} : y \mapsto K_0 \cdots K_{n-1}(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) + iy), \quad 0 \leq y \leq h'_n$$

of the curves $\gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ converge to the parametrizations

$$\phi_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)} : y \mapsto K_0 \cdots K_{n-1}(x(\epsilon_0, \dots, \epsilon_{n-2}, 0) + iy), \quad 0 \leq y \leq h'_n$$

of the curves $\gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)}$ in all C^r norms as $a_n \rightarrow \infty$; it follows then from the following standard fact, which we state here without proof, that the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}$ also converge in all C^r norms to the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)}$ as $a_n \rightarrow \infty$:

Lemma 2.3.4. *Let $\phi_n : y \mapsto u_n(y) + iv_n(y)$ and $\phi : y \mapsto u(y) + iv(y)$, defined for $0 \leq y \leq h$, be C^∞ parametrizations of smooth curves, such that $v'_n(y) > 0$, $v'(y) > 0$ for all n, y , and $v_n(0), v(0) < A < B < v_n(h), v(h)$ for all n for fixed constants A and B . We define the C^∞ parametrizations σ_n, σ by*

$$\begin{aligned} \sigma_n(t) &= u_n \circ v_n^{-1}(t) + it, \quad v_n(0) \leq t \leq v_n(h) \\ \sigma(t) &= u \circ v^{-1}(t) + it, \quad v(0) \leq t \leq v(h) \end{aligned}$$

Suppose $\|\phi_n - \phi\|_{C^r[0, h]} \rightarrow 0$ as $n \rightarrow \infty$ for all $r \geq 0$. Then $\|\sigma_n - \sigma\|_{C^r[A, B]} \rightarrow 0$ as $n \rightarrow \infty$ for all $r \geq 0$.

It follows that by choosing a_n large enough, we can have

$$\|\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})} - \sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)}\|_{C^n} \leq \frac{1}{2^n}$$

for all $\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}$.

In condition (e) below we will require a further control on the C^0 distance between these parametrizations, but for now it follows from the above condition on a_n that for any $\tilde{\epsilon} = (\epsilon_0, \epsilon_1, \dots) \in \{-1, 0, 1\}^{\mathbf{N}}$, the sequence $\{\sigma_{(\epsilon_0, \dots, \epsilon_m)}\}_{m \geq 0}$ will be Cauchy in every C^r norm on $[1, 2]$, and hence convergent in every C^r norm to a parametrization $\sigma_{\tilde{\epsilon}}$ of a C^∞ curve $\Gamma_{\tilde{\epsilon}}$. Thus we are guaranteed that the map $\Phi : (\theta(\tilde{\epsilon}), t) \mapsto \sigma_{\tilde{\epsilon}}(t)$ is, firstly, well-defined, and secondly, smooth in t for fixed $\theta(\tilde{\epsilon}) \in C_0$, as stated in the Main Theorem.

We state the further conditions required on a_n so that in addition Φ is a bi-Hölder homeomorphism:

(d) The following condition will be used in section 3 to prove that the map Φ is α -Hölder for every α , $0 < \alpha < 1$:

We would like to choose a_n large enough so that we have an estimate of the form

$$|\Phi(\theta, t) - \Phi(\theta_0, t)| \leq \left(\frac{1}{a_0 \cdots a_n} \right)^{\alpha_n}$$

for all $t \in [1, 2]$, $\theta_0 = \theta(\epsilon_0, \dots, \epsilon_{n-2})$, $\theta = \theta(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) \in C_0$ s.t. $|\theta - \theta_0| = \frac{1}{a_0 \cdots a_n}$.

So fix $t_0 \in [1, 2]$, and let $z_0 = x(\epsilon_0, \dots, \epsilon_{n-2}, 0) + iy_0 = x_0 + iy_0$ and $z = x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) + iy = x + iy$ be points s.t.

$$\begin{aligned}\Phi(\theta_0, t_0) &= K_0 \cdots K_{n-1}(z_0), \\ \Phi(\theta, t_0) &= K_0 \cdots K_{n-1}(z).\end{aligned}$$

Now

$$\frac{\partial v_{n-1}}{\partial y}(x_0 + iy_0) > 0 \text{ (by the induction hypotheses),}$$

so by the Implicit Function Theorem, the equation

$$v_{n-1}(x + iy) = t$$

determines, for (x, t) in a neighbourhood of (x_0, t_0) , y as a smooth function of (x, t) (here we consider x_0, t_0, y_0 as fixed, and x, t as variable, since x depends on a_n which we are varying), say $y = y(x, t)$, such that $y(x_0, t_0) = y_0$. Thus

$$\Phi(\theta, t) = u_{n-1}(x + iy(x, t)) + it$$

is a smooth function of (x, t) in a neighbourhood of (x_0, t_0) . Since this holds for all (x_0, y_0) , it follows that $\Phi(\theta, t)$ is a smooth function of (x, t) , $\Phi(\theta, t) = f(x, t)$ say, in a compact neighbourhood of the compact $\{(x, t) : x = x(\epsilon_0, \dots, \epsilon_{n-2}, 0), \epsilon_0, \dots, \epsilon_{n-2} \in \{-1, 0, 1\}, t \in [1, 2]\}$. Hence given $t \in [1, 2]$ and $\theta_0, \theta \in C_0$ as above with $|\theta - \theta_0| = \frac{1}{a_0 \cdots a_n}$, we can estimate

$$|\Phi(\theta, t) - \Phi(\theta_0, t)| = |f(x, t) - f(x_0, t)| \leq \|f\|_{C^1} |x - x_0| = \|f\|_{C^1} \cdot \frac{1}{a_n}$$

We note here that the constant $\|f\|_{C^1}$ only depends on the maps K_0, \dots, K_{n-1} and the parameters a_0, \dots, a_{n-1} already chosen, and not on a_n . Since $\alpha_n < 1$, it follows that by taking a_n large enough we can ensure that

$$\begin{aligned}|\Phi(\theta, t) - \Phi(\theta_0, t)| &\leq \|f\|_{C^1} \cdot \frac{1}{a_n} \\ &\leq \left(\frac{1}{a_0 \cdots a_{n-1} a_n} \right)^{\alpha_n}\end{aligned}$$

(e) The following condition will be used in section 3 to prove that the the map Φ is injective and Φ^{-1} Lipschitz:

Applying Lemma 2.3.1 to $K_n = \frac{1}{a_n} K_{\delta(a_n, h_n)}$, with $v = h_n - 1/2 < h = h_n$ and $a = a_n$, we require that a_n is chosen large enough so that

$$|a_n \cdot K'_n(z)| > 1 \text{ for all } z = x + iy, x \in \mathbf{Z}, 0 \leq y \leq Y_{n+1}$$

where Y_{n+1} depending on a_n is chosen such that $\text{Im } K_n(iY_{n+1}) = h_n - 1/2$.

Since the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}$ converge uniformly to the parametrizations $\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)}$ when $a_n \rightarrow +\infty$, for a_n large enough the curves $\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}$ are contained in the neighbourhoods $K_0 \dots K_{n-1}(U(\epsilon_0, \dots, \epsilon_{n-2}))$ of the curves $\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2})}$. We require that a_n is chosen large enough so that

$$\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})} \subset K_0 \dots K_{n-1}(U(\epsilon_0, \dots, \epsilon_{n-2}))$$

for all $\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}$ (for $n = 1$ we require that $\Gamma_{(\epsilon_0)} \subset K_0(U)$, $\epsilon_0 \in \{-1, 0, 1\}$). We also require that

$$\|\sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})} - \sigma_{(\epsilon_0, \dots, \epsilon_{n-2}, 0)}\|_{C^0} \leq \frac{1/20}{a_0 \dots a_{n-1}}$$

for all $\epsilon_0, \dots, \epsilon_{n-1} \in \{-1, 0, 1\}$ (note that a_0, \dots, a_{n-1} have already been chosen and fixed).

(f) For the hedgehog to be non-linearisable :

Let P_n be the image under K_n of the upper ends of the 1-cylinders in $\mathcal{S}_{\delta_{n+1}}$, or more precisely,

$$P_n := z_n + \frac{1}{a_n} \mathbf{Z}$$

where z_n is the limit of $K_n(z)$ as $\text{Im } z \rightarrow +\infty$ in one of the 1-cylinders of $\mathcal{S}_{\delta_{n+1}}$ (note P_n is independent of the choice of 1-cylinder, since their images differ by $1/a_n$ translations), and define the sets

$$\begin{aligned} O_{n,n+1} &:= P_n \\ O_{n-1,n+1} &:= K_{n-1}(P_n) \\ O_{n-2,n+1} &:= K_{n-2}K_{n-1}(P_n) \\ &\dots \\ O_{0,n+1} &:= K_0 \dots K_{n-1}(P_n) \end{aligned}$$

We have:

Proposition 2.3.5 *For all $0 \leq i \leq n+1, m \geq n+1$, for every $F \in \mathcal{F}_{i,m}$, the set $O_{i,n+1}$ is a union of periodic orbits (modulo \mathbf{Z}).*

Proof :

Fix $m \geq n+1$. Consider an $F \in \mathcal{F}_{n,m}$. The map F is the conjugate under K_n of a lift \tilde{F}_t of a map in $\mathcal{F}_{n+1,m}$ for a time $t \in \mathbf{Z} + [-2a_{n+1}^{-1}, 2a_{n+1}^{-1}]$, say $t = l + \alpha, l \in \mathbf{Z}, \alpha \in [-2a_{n+1}^{-1}, 2a_{n+1}^{-1}]$. Suppose $\text{Im } z \rightarrow +\infty$ in the 1-cylinder of $\mathcal{S}_{\delta_{n+1}}$ whose ramification point lies at a point $q \in \mathbf{Z}$ in the 0-sheet; then $\tilde{F}_t(z)$ escapes to infinity through the 1-cylinder whose ramification point lies at the point $q+l$ in the 0-sheet, and $\text{Im } \tilde{F}_t(z) \rightarrow +\infty$. Since P_n consists of the images under K_n of the upper ends of the 1-cylinders, it follows that the orbits under F of points of $O_{n,n+1} = P_n$ are periodic (modulo \mathbf{Z}) and contained in $O_{n,n+1}$.

For $0 \leq i \leq n-1$, since the elements of $\mathcal{F}_{i,m}$ are given by conjugating those of $\mathcal{F}_{n,m}$ by the map $K_i \dots K_{n-1}$, and $O_{i,n+1}$ is given by transporting $O_{n,n+1}$ by the same map, the desired conclusion follows. \diamond .

Using property **(5)** of the maps $\frac{1}{a}K_\delta$, we choose a_n large enough so that

$$P_n \subseteq \{ \operatorname{Im} z \geq h_n - 1 \}.$$

The induction hypothesis **1.** then guarantees that $O_{0,n+1}$, the image of P_n under $K_0 \dots K_{n-1}$, which is a set of periodic orbits (modulo \mathbf{Z}) for all the semi-flows $\mathcal{F}_{0,m}$, $m \geq n+1$, will be at a height at least $h_0 + n$ in the upper half-plane, so the semi-flows $(\mathcal{F}_{0,k})_{k \geq 0}$ will have non-linearities at increasing heights.

We now choose and fix $a_n \geq 5$ large enough to satisfy all the conditions **(a)-(f)** above. Thus the map K_n is determined.

To complete the induction step, it remains to choose the constants h_{n+1} , h'_{n+1} , τ_{n+1} and the domain D_{n+1} so that the induction hypotheses are satisfied for the next step.

For hypothesis **1.** :

Now K_n is fixed, so using property **(4)** of the maps $\frac{1}{a}K_\delta$, we choose $h_{n+1} > 2$ such that $\operatorname{Im} K_0 \dots K_n(x + i(h_{n+1} - 1)) = v_n(x + i(h_{n+1} - 1)) > h_0 + (n + 1)$.

For hypothesis **2.** :

Using property **(4)** of the maps $\frac{1}{a}K_\delta$, we choose $h'_{n+1} > 0$ such that

$$\operatorname{Im} K_n(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + ih'_{n+1}) = h'_n$$

(note $x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0)$'s are integers).

It follows from the induction hypothesis **2.** and condition **(b)** on a_n that

$$\begin{aligned} v_n(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + i0) &= v_{n-1}(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) + i0) < 1, \\ v_n(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + ih'_{n+1}) &= v_{n-1}(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) + ih'_n) > 2 \text{ and} \\ \frac{\partial v_n}{\partial y} &> 0 \end{aligned}$$

on all segments $[x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + i0, x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + ih'_{n+1}]$

by the Chain rule, since the $x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0)$'s are integers, so on these vertical segments we have $K'_n(z)$ is real and positive (by properties **(2)**, **(3)** of the maps $\frac{1}{a}K_\delta$), and hence

$$\frac{\partial v_n}{\partial y} = \frac{\partial v_{n-1}}{\partial y} \cdot K'_n(x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + iy) > 0.$$

So by continuity and compactness of the above line segments, we choose $\tau_{n+1} > 0$ small enough such that

$$\begin{aligned} v_n(x + i0) < 1, v_n(x + ih'_{n+1}) > 2 \quad \text{and} \\ \frac{\partial v_n}{\partial y}(x + iy) > 0 \\ \text{for } |x - x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0)| \leq \tau_{n+1}, \quad 0 \leq y \leq h'_{n+1}, \\ \text{and } \epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}. \end{aligned}$$

For hypothesis **3.**:

Using condition **(e)** above on a_n , compactness of the segments $[x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + i0, x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0) + iY_{n+1}]$ and continuity, we also choose τ_{n+1} small enough so that for all $\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}$ we have

$$|a_n \cdot K'_n(z)| > 1 \quad \text{for all } z = x + iy, |x - x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0)| \leq \tau_{n+1}, \quad 0 \leq y \leq Y_{n+1}$$

For hypothesis **4.**:

We know from condition **(e)** above on a_n that

$$\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})} \subset K_0 \dots K_{n-1}(U(\epsilon_0, \dots, \epsilon_{n-2}))$$

for all $\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}$, which implies that

$$(K_0 \dots K_{n-1})^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}) \subset U(\epsilon_0, \dots, \epsilon_{n-2}).$$

In addition we have

$$(K_0 \dots K_{n-1})^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}) \subset K_n(\{z = x + iy, |x - x(\epsilon_0, \dots, \epsilon_{n-1}, 0)| \leq \tau_{n+1}, 0 \leq y \leq Y_{n+1}\}).$$

Since each set $(K_0 \dots K_{n-1})^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})})$ is in fact a compact vertical line segment (contained in the half-line $\xi_{x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}$), we can choose open rectangles $R(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})$ such that

$$(K_0 \dots K_{n-1})^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})}) \subset R(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})$$

and

$$R(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) \subset U(\epsilon_0, \dots, \epsilon_{n-2}),$$

$$R(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) \subset K_n(\{z = x + iy, |x - x(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}, 0)| \leq \tau_{n+1}, 0 \leq y \leq Y_{n+1}\}).$$

We define $U(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1})$ by

$$U(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}) := K_n^{-1}(R(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}))$$

and D_{n+1} by

$$D_{n+1} := \bigcup_{\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1} \in \{-1, 0, 1\}} U(\epsilon_0, \dots, \epsilon_{n-2}, \epsilon_{n-1}).$$

We observe that with these definitions all the conditions of the induction hypothesis **4.** are satisfied.

Thus the induction hypotheses **1.** to **4.** required for stage $n + 1$ are satisfied and the induction can continue.

This ends the inductive choice of the parameters.

3. Proof of the main theorem.

In the course of the induction above we noted that we get a well defined map $\Phi : C_0 \times [1, 2] \rightarrow \mathcal{C}$, which is C^∞ smooth in $t \in [1, 2]$ for fixed $\theta \in C_0$. It remains to prove that Φ is a homeomorphism, Φ is α -Hölder regular for all α , $0 < \alpha < 1$, Φ^{-1} is Lipschitz, and the elements f_t of the limit semi-flow are non-linearisable for irrational t .

We begin with a lemma which will be a basic tool for the proofs which follow.

Lemma 3.1 *Suppose $\tilde{\epsilon}^{(1)}, \tilde{\epsilon}^{(2)} \in \{-1, 0, 1\}^{\mathbf{N}}$ are such that*

$$\tilde{\epsilon}_n^{(1)} = \tilde{\epsilon}_n^{(2)} \text{ for } n < i_0, \text{ and } \tilde{\epsilon}_n^{(1)} < \tilde{\epsilon}_n^{(2)} \text{ for } n = i_0$$

for some integer $i_0 \geq 0$. Then

$$\theta(\tilde{\epsilon}^{(1)}) \leq \theta + \frac{1}{3}\delta < \theta + \frac{2}{3}\delta \leq \theta(\tilde{\epsilon}^{(2)})$$

where

$$\theta = \theta(\epsilon_0^{(1)}, \dots, \epsilon_{i_0}^{(1)}, 0, 0, \dots) \text{ and } \delta = \frac{1}{a_0 a_1 \dots a_{i_0+1}}.$$

Proof :

$$\begin{aligned} \theta(\tilde{\epsilon}^{(1)}) &= \frac{\epsilon_0^{(1)}}{a_0 a_1} + \dots + \frac{\epsilon_{i_0}^{(1)}}{a_0 \dots a_{i_0+1}} + \frac{\epsilon_{i_0+1}^{(1)}}{a_0 \dots a_{i_0+1} a_{i_0+2}} + \dots \\ &\leq \left(\frac{\epsilon_0^{(1)}}{a_0 a_1} + \dots + \frac{\epsilon_{i_0}^{(1)}}{a_0 \dots a_{i_0+1}} \right) + \frac{1}{a_0 \dots a_{i_0+1}} \left(\frac{1}{a_{i_0+2}} + \frac{1}{a_{i_0+2} a_{i_0+3}} + \dots \right) \\ &\leq \theta + \delta \left(\frac{1}{4} + \frac{1}{4^2} + \dots \right) \\ &= \theta + \frac{1}{3}\delta \end{aligned}$$

Similarly, noting that $\epsilon_{i_0}^{(2)} \geq \epsilon_{i_0}^{(1)} + 1$, we get

$$\theta(\tilde{\epsilon}^{(2)}) \geq \theta + \frac{2}{3}\delta \quad \diamond$$

The following lemma will allow us to approximate $\Phi(\theta, t)$, where $\theta = \theta(\epsilon_0, \epsilon_1, \dots)$, by $\Phi(\theta_n, t)$, where $\theta_n = \theta(\epsilon_0, \dots, \epsilon_n)$, and hence to use the estimates obtained at finite stages in the construction in section 2.3.

Lemma 3.2 *Let $\theta = \theta(\epsilon_0, \epsilon_1, \dots)$ and let $\theta_n = \theta(\epsilon_0, \dots, \epsilon_n)$ for all $n \geq 0$. Then*

$$|\Phi(\theta, t) - \Phi(\theta_n, t)| \leq \frac{1/10}{a_0 \dots a_{n+1}}$$

Proof:

Using the estimate in (e) of the inductive construction we have

$$\begin{aligned} |\Phi(\theta, t) - \Phi(\theta_n, t)| &\leq \sum_{k=n+1}^{\infty} |\Phi(\theta_k, t) - \Phi(\theta_{k-1}, t)| \\ &\leq \sum_{k=n+1}^{\infty} \frac{1/20}{a_0 \dots a_k} \\ &\leq \frac{1/20}{a_0 \dots a_{n+1}} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots \right) \\ &\leq \frac{1/10}{a_0 \dots a_{n+1}} \end{aligned}$$

We can now prove

Proposition 3.3 *Φ is α -Hölder for every α , $0 < \alpha < 1$.*

We first prove

Lemma 3.4 *Φ is α -Hölder in $\theta \in C_0$ for every fixed $t \in [1, 2]$, with a uniform Hölder constant independent of t .*

Proof :

Let α be given, $0 < \alpha < 1$. Choose $N \geq 1$ such that $\alpha_n > \alpha$ for $n \geq N$.

Fix $t \in [1, 2]$ and let $\theta, \theta' \in C_0, \theta \neq \theta'$, say $\theta = \theta_{\tilde{\epsilon}}$ and $\theta' = \theta_{\tilde{\epsilon}'}$. We note, using condition (e) of the construction and the fact that Φ takes values in the strip S , that

$$|\operatorname{Re} \Phi| \leq \sum_{n=1}^{\infty} \frac{1/20}{a_0 \dots a_{n-1}} < 1/2, \quad 1 \leq \operatorname{Im} \Phi \leq 2$$

thus Φ is a bounded function and it suffices to prove a Hölder estimate on Φ for all θ, θ' sufficiently close to each other. So we assume that $\tilde{\epsilon}$ and $\tilde{\epsilon}'$ agree up to $(n-1)$ places with $n \geq N$, ie $\epsilon_i = \epsilon'_i$, $0 \leq i \leq n-2$, $\epsilon_{n-1} \neq \epsilon'_{n-1}$, say $\epsilon_{n-1} < \epsilon'_{n-1}$.

Then $\theta' - \theta \geq \frac{1}{3}\delta$, where $\delta = \frac{1}{a_0 \dots a_n}$. For $k \geq 0$, we let $\theta_k = \theta(\epsilon_0, \dots, \epsilon_k), \theta'_k = \theta(\epsilon'_0, \dots, \epsilon'_k)$. Using the estimate in **(d)** of the inductive construction and Lemma 3.2 above, we have

$$\begin{aligned}
|\Phi(\theta', t) - \Phi(\theta, t)| &\leq |\Phi(\theta'_{n-1}, t) - \Phi(\theta_{n-1}, t)| + |\Phi(\theta', t) - \Phi(\theta'_{n-1}, t)| \\
&\quad + |\Phi(\theta, t) - \Phi(\theta_{n-1}, t)| \\
&\leq |\Phi(\theta'_{n-1}, t) - \Phi(\theta_{n-2}, t)| + |\Phi(\theta_{n-1}, t) - \Phi(\theta_{n-2}, t)| \\
&\quad + |\Phi(\theta', t) - \Phi(\theta'_{n-1}, t)| + |\Phi(\theta, t) - \Phi(\theta_{n-1}, t)| \\
&\leq \delta^{\alpha_n} + \delta^{\alpha_n} + |\Phi(\theta', t) - \Phi(\theta'_{n-1}, t)| + |\Phi(\theta, t) - \Phi(\theta_{n-1}, t)| \\
&\leq 2\delta^{\alpha_n} + \frac{1}{10}\delta + \frac{1}{10}\delta \\
&\leq 3\delta^\alpha \\
&\leq 3 \cdot 3^\alpha \cdot |\theta' - \theta|^\alpha
\end{aligned}$$

(note that the Hölder constant appearing here is independent of t) \diamond .

Proof of proposition 3.3 :

We first note that for all $\tilde{\epsilon} \in \{-1, 0, 1\}^{\mathbb{N}}$, the parametrizations $\sigma_{\tilde{\epsilon}}$ have uniformly bounded derivatives, since

$$\begin{aligned}
\|\sigma_{\tilde{\epsilon}}\|_{C^1} &\leq \|\sigma_{(0,0,0,\dots)}\|_{C^1} + \|\sigma_{\tilde{\epsilon}} - \sigma_{(0,0,0,\dots)}\|_{C^1} \\
&\leq \|\sigma_{(0,0,0,\dots)}\|_{C^1} + \left(\frac{1}{2} + \frac{1}{2^2} + \dots\right) \\
&= \|\sigma_{(0,0,0,\dots)}\|_{C^1} + 1 = M = \text{constant}
\end{aligned}$$

So, with the same notation as in Lemma 3.4 above, for points $(\theta', t'), (\theta, t)$ with t' not necessarily equal to t , using Lemma 3.4 we have

$$\begin{aligned}
|\Phi(\theta', t') - \Phi(\theta, t)| &\leq |\Phi(\theta', t') - \Phi(\theta, t')| + |\Phi(\theta, t') - \Phi(\theta, t)| \\
&\leq 3 \cdot 3^\alpha \cdot |\theta' - \theta|^\alpha + \|\sigma_{\tilde{\epsilon}}\|_{C^1} |t' - t| \\
&\leq (3 \cdot 3^\alpha + M) \cdot |(\theta', t') - (\theta, t)|^\alpha \quad \diamond
\end{aligned}$$

Proposition 3.5 Φ is injective, and Φ^{-1} is Lipschitz.

We need the following lemma:

Lemma 3.6 For $z, z' \in U(\epsilon_0, \dots, \epsilon_{n-1}, \epsilon_n)$, we have

$$|K_0 \dots K_n(z) - K_0 \dots K_n(z')| \geq \frac{|z - z'|}{a_0 \dots a_n}$$

Proof :

Recalling the nesting condition of the induction hypothesis 4.,

$$K_{k+1}(U(\epsilon_0, \dots, \epsilon_{n-1}, \epsilon_k)) \subset U(\epsilon_0, \dots, \epsilon_{k-1}), \quad k \geq 1$$

(the condition is $K_1(U(\epsilon_0)) \subset U$ for $k = 0$), we can apply Proposition 2.3.2 repeatedly to obtain

$$\begin{aligned} |K_0 \dots K_n(z_1) - K_0 \dots K_n(z_2)| &\geq \frac{|K_1 \dots K_n(z_1) - K_1 \dots K_n(z_2)|}{a_0} \\ &\geq \frac{|K_2 \dots K_n(z_1) - K_2 \dots K_n(z_2)|}{a_0 a_1} \\ &\dots \\ &\geq \frac{|z_1 - z_2|}{a_0 \dots a_n}. \end{aligned}$$

◇

Proof of Proposition 3.5:

Let $s + it = \Phi(\theta, t)$, $s' + it' = \Phi(\theta', t') \in \mathcal{C}$, for $(\theta, t), (\theta', t') \in C_0 \times [1, 2]$, with $\theta = \theta(\tilde{\epsilon})$ and $\theta' = \theta(\tilde{\epsilon}')$ say, $\theta \neq \theta'$. Let $n \geq 0$ be such that $\epsilon_i = \epsilon'_i$ for $i \leq n-1$, and $\epsilon_n \neq \epsilon'_n$. Then we have seen before that $|\theta - \theta'| \geq \frac{1}{3}\delta$, where $\delta = 1/(a_0 \dots a_{n+1})$; we note also however that

$$\begin{aligned} |\theta - \theta'| &\leq \frac{|\epsilon_n - \epsilon'_n|}{a_0 \dots a_{n+1}} + \frac{|\epsilon_{n+1} - \epsilon'_{n+1}|}{a_0 \dots a_{n+1} a_{n+2}} + \dots \\ &\leq \frac{2}{a_0 \dots a_{n+1}} \left(1 + \frac{1}{4} + \frac{1^2}{4} + \dots \right) \\ &\leq \frac{8}{3}\delta \end{aligned}$$

For $k \geq 0$ we let

$$\begin{aligned} \theta_k &= \theta(\epsilon_0, \dots, \epsilon_{k-1}, \epsilon_k), \quad \theta'_k = \theta(\epsilon'_0, \dots, \epsilon'_{k-1}, \epsilon'_k), \\ \Phi(\theta_k, t) &= s_k + it, \quad \Phi(\theta'_k, t') = s'_k + it'. \end{aligned}$$

We let z_n, z'_n be points such that

$$\begin{aligned} K_0 \dots K_n(z_n) &= \Phi(\theta_n, t) = s_n + it \\ K_0 \dots K_n(z'_n) &= \Phi(\theta'_n, t') = s'_n + it' \end{aligned}$$

We observe, using condition (e) on a_{n+1} , that

$$z_n \in (K_0 \dots K_n)^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1}, \epsilon_n)}) \subset U(\epsilon_0, \dots, \epsilon_{n-1}).$$

and

$$z'_n \in (K_0 \dots K_n)^{-1}(\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1}, \epsilon'_n)}) \subset U(\epsilon_0, \dots, \epsilon_{n-1}).$$

Hence applying Lemma 3.6 above gives

$$|(s_n + it) - (s'_n + it')| = |K_0 \dots K_n(z_n) - K_0 \dots K_n(z'_n)| \geq \frac{|z_n - z'_n|}{a_0 \dots a_n}$$

Also, by Lemma 3.2 we have

$$\begin{aligned} |(s + it) - (s_n + it)| &= |\Phi(\theta, t) - \Phi(\theta_n, t)| \\ &\leq \frac{1/10}{a_0 \dots a_{n+1}} = 1/10 \cdot \delta \end{aligned}$$

Similarly $|(s' + it') - (s'_n + it')| \leq 1/10 \cdot \delta$. Therefore

$$\begin{aligned} |(s + it) - (s' + it')| &\geq |(s_n + it) - (s'_n + it')| - \frac{1}{10}\delta - \frac{1}{10}\delta \\ &\geq \frac{|z_n - z'_n|}{a_0 \dots a_n} - \frac{1}{5}\delta \\ &\geq \frac{|x_n - x'_n|}{a_0 \dots a_n} - \frac{1}{5}\delta \\ &\geq \frac{1}{a_0 \dots a_n a_{n+1}} - \frac{1}{5}\delta \\ &= \frac{4}{5}\delta \\ &\geq \frac{4}{5} \cdot \frac{3}{8} |\theta - \theta'| \end{aligned}$$

Combining this with $|(s + it) - (s' + it')| \geq |t - t'|$ gives

$$\begin{aligned} |(\theta, t) - (\theta', t')| &\leq |\theta - \theta'| + |t - t'| \\ &\leq \frac{5}{4} \cdot \frac{8}{3} |(s + it) - (s' + it')| + |(s + it) - (s' + it')| \\ &= \frac{13}{3} |(s + it) - (s' + it')| = \frac{13}{3} |\Phi(\theta, t) - \Phi(\theta', t')| \end{aligned}$$

(note that this inequality also holds when $\theta = \theta'$, since then $|(\theta, t) - (\theta', t')| = |t - t'| \leq \frac{13}{3}|t - t'| = \frac{13}{3}|\Phi(\theta, t) - \Phi(\theta', t')|$).

Thus Φ is injective and Φ^{-1} is Lipschitz. \diamond

In particular, Φ and Φ^{-1} are continuous, and hence we have

Corollary 3.7 Φ is a homeomorphism. Hence the sets \mathcal{C} and $\mathcal{E} = E(\mathcal{C})$ are combs.

Remark :

In the course of proving Lemma 3.4, we observed that $|\operatorname{Re} \Phi| < 1/2$, so that \mathcal{C} is contained in the strip $\{-1/2 < \operatorname{Re} z < 1/2\}$, which we recall was enough to guarantee that all properties of Φ also hold for Ψ .

Proposition 3.8 *The comb \mathcal{C} is contained in the hedgehog \mathcal{H} .*

Proof:

For each curve $\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$, we know that

$$\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})} \subset K_0 K_1 \dots K_{n-1}(\xi_{x(\epsilon_0, \dots, \epsilon_{n-1})}) = K_0 K_1 \dots K_{n-1} K_n(\xi_{a_n \cdot x(\epsilon_0, \dots, \epsilon_{n-1})})$$

Since $a_n \cdot x(\epsilon_0, \dots, \epsilon_{n-1})$ is an integer, it follows from Proposition 2.2.1 that each curve $\Gamma_{(\epsilon_0, \dots, \epsilon_{n-1})}$ is contained in \mathcal{H} . Since curves in \mathcal{C} are uniform limits of such curves and \mathcal{H} is closed it follows that \mathcal{C} is contained in \mathcal{H} . \diamond

Finally, we observe that the hedgehog is non-linearisable:

Proposition 3.9 *Each f_t of the limit semi-flow is non-linearisable for $t \in C \cap (\mathbf{R} - \mathbf{Q})$.*

Proof:

From Proposition 2.3.5, we know that the set $O_{0,n+1}$ forms a set of periodic orbits (modulo \mathbf{Z}) for all elements of the semi-flows $\mathcal{F}_{0,m}$ for all $m \geq n+1$. We observe also that for all $m \geq n+1$ and all $F \in \mathcal{F}_{0,m}$, the periods of these orbits are uniformly bounded above by the cardinal of $O_{0,n+1}/\mathbf{Z}$ (which is in fact equal to a_n). Thus these orbits are also periodic orbits (modulo \mathbf{Z}) for elements of the limit semi-flow \mathcal{F} , and each $O_{0,n+1}$ forms a set of periodic orbits (modulo \mathbf{Z}) for all elements of \mathcal{F} . By **(f)** of the inductive construction, P_n is contained in the half-plane $\{\operatorname{Im} z \geq h_n - 1\}$, and hence by the induction hypothesis **1.**, $O_{0,n+1}$ is contained in the half-plane $\{\operatorname{Im} z \geq h_0 + n\}$. Thus each f_t has a sequence of periodic orbits $E(O_n)$ which accumulate 0, and is hence non-linearisable for irrational $t \in C \cap (\mathbf{R} - \mathbf{Q})$. \diamond

This ends the proof of the main theorem.

4. Remarks.

1. In the construction given here, we make the heights of the periodic orbits $O_{0,n+1}$ increase to infinity, so that the hedgehog is non-linearisable. If instead the heights h_n are chosen differently so that the heights of the orbits $O_{0,n+1}$ do not escape to infinity but converge to a finite height, then the limit dynamics is linearisable above this height, and we obtain a linearisable hedgehog containing a Siegel disk (in fact the construction in [PM4] is carried out this way). It is possible in this case too to construct smooth combs inside the hedgehog, and to make sure that they are non-trivial, lying outside the Siegel disk.

2. We note that though the hedgehog is invariant under the dynamics, the comb itself is not.

3. Since the comb \mathcal{C} is bi-Hölder- α equivalent to the comb $C_0 \times [1, 2]$ for all α , $0 < \alpha < 1$, \mathcal{C} has the same Hausdorff dimension as $C_0 \times [1, 2]$, which is the dimension of C_0 plus one. It is easy to see from the explicit representation of C_0 in terms of the a_n 's that C_0 can be made zero-dimensional by taking the a_n 's growing fast enough. Thus we can ensure that the comb \mathcal{C} has Hausdorff dimension one. We can also get combs of Hausdorff dimension one by taking small enough Cantor subsets $C'_0 \subset C_0$ of C_0 , with C'_0 zero-dimensional.

4. The rotation numbers $t \in \mathcal{C}$ for the limit dynamics include numbers of the form

$$t = \sum_{n=0}^{\infty} \frac{\epsilon_n}{a_0 a_1 \dots a_n}$$

where $\epsilon_n \in \{-1, 0, 1\}$, $n \geq 0$. As the construction requires the a_n 's to be growing rapidly, this means that we obtain very Liouville rotation numbers.

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References

- [Bi] G.D.BIRKHOFF, *Surface transformations and their dynamical applications*, Acta Mathematica, **43**, (1920). Also, Collected Mathematical Papers II, p.111
- [Cr1] H.CREMER, *Zum Zentrumproblem*, Math Ann., vol 98, 1928, p. 151-153
- [Cr2] H.CREMER, *Über die Häufigkeit der Nichtzentren*, Math Ann., vol 115, 1938, p.573-580
- [PM1] R.PEREZ-MARCO, *Fixed points and circle maps*, Acta Mathematica, **179:2**, 1997, p.243-294.
- [PM2] R.PEREZ-MARCO, *Topology of Julia sets and hedgehogs*, Preprint, Université de PARIS-SUD, 1994.
- [PM3] R.PEREZ-MARCO, *Uncountable number of symmetries for non-linearisable holomorphic dynamics*, Inventiones Mathematicae, **119**, **1**, p.67-127,1995; (C.R. Acad. Sci. Paris, **313**, 1991, p. 461-464).
- [PM4] R.PEREZ-MARCO, *Siegel disks with smooth boundary*, Preprint, Université de PARIS-SUD, 1997, www.math.ucla.edu/~ricardo, submitted to the Annals of Mathematics in 1997.
- [PM5] R.PEREZ-MARCO, *Hedgehog's Dynamics*, Preprint, UCLA, 1996.
- [PM6] R.PEREZ-MARCO, *Sur les dynamiques holomorphes non linéarisables et une conjecture de V.I. Arnold*, Ann. Scient. Ec. Norm. Sup. 4 serie, **26**, 1993, p.565-644; (C.R. Acad. Sci. Paris, **312**, 1991, p.105-121).

[Si] C.L.SIEGEL, *Iteration of Analytic Functions*, Ann. Math., vol.43, 1942, p. 807-812

[Y] J.C.YOCCOZ, *Petits diviseurs en dimension 1*, S.M.F., Asterisque **231** (1995)