CANNON-THURSTON MAPS FOR KLEINIAN GROUPS

MAHAN MJ

Abstract. We show that Cannon-Thurston maps exist for degenerate free groups without parabolics, i.e. for handlebody groups. Combining these techniques with earlier work proving the existence of Cannon-Thurston maps for surface groups, we show that Cannon-Thurston maps exist for arbitrary finitely generated Kleinian groups without parabolics, proving a conjecture of McMullen. We also show that point pre-images under Cannon-Thurston maps for degenerate free groups without parabolics correspond to end-points of leaves of an ending lamination in the Masur domain, proving a conjecture of Otal.

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Contents

1. Introduction

In [28] we showed that simply or doubly degenerate surface Kleinian groups without accidental parabolics admit Cannon-Thurston maps, answering affirmatively a question of Cannon and Thurston (Section 6 of [10] [11]). In [29] we had shown that point pre-images of the Cannon-Thurston map for simply or doubly degenerate groups without parabolics correspond to endpoints of leaves of ending laminations. The main aim of this paper is to extend these results to arbitrary finitely generated

Kleinian groups without parabolics. The principal new ingredient is a proof of the existence of Cannon-Thurston maps for degenerate handlebody groups.

The following is the main theorem of this paper.

Theorem 3.5 *Let G be a finitely generated free degenerate Kleinian group without parabolics. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$. Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma$ of Γ. Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial\Gamma$ if and only if a, b are either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon.*

The proof of Theorem 3.5 generalizes with minor modifications to arbitrary finitely generated Kleinian group without parabolics.

Theorems 2.17 and 3.6 *Let G be a finitely generated Kleinian group without parabolics. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$. Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma$ of Γ. Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial\Gamma$ if and only if a, b are either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon.*

In both Theorem 3.5 and 2.17 ending laminations corresponding to an end bounded by a compressible surface are regarded as elements of the Masur domain. For ease of exposition, throughout this paper, we shall first work out the problem for free groups and then indicate the generalization to arbitrary finitely generated Kleinian groups without parabolics.

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1.1. **Relative Hyperbolicity.** We refer the reader to Farb [13] for terminology and details on relative hyperbolicity and electric geometry.

Let X be a δ-hyperbolic metric space, and $\mathcal{H}$ a family of C-quasiconvex, D-separated, collection of subsets. Then by work of Farb [13], X_{el} obtained by electrocuting the subsets in $\mathcal{H}$ is a $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. (Recall [13] that **electrocution** means constructing an auxiliary space $X_{el} = X \bigcup_{H\in\mathcal{H}}(H \times I)$ with $H \times \{0\}$ identified to $H \subset X$ and $H \times \{1\}$ equipped with the zero metric. This is a geometric 'coning' construction.)

We shall say that a (quasi)geodesic does not *backtrack* if it does not re-enter any $H \in \mathcal{H}$ after leaving it.

Electric P-quasigeodesics without backtracking are said to have similar intersection patterns if for β, γ electric P-quasigeodesics without backtracking both joining x, y, the following are satisfied.

(1) *Similar Intersection Patterns 1:* if precisely one of $\{\beta, \gamma\}$ meets an ϵ-neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point to the exit point is at most D.

(2) *Similar Intersection Patterns 2:* if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point of β to that of γ is at most D; similarly for exit points.

Definition 1.1. [13] [2] *Let X be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets such that X is* weakly hyperbolic *relative to the collection $\mathcal{H}$, i.e. X_{el} is a hyperbolic metric space. If any pair of P- electric quasigeodesics without backtracking starting and ending at the same point have* similar intersection patterns *with elements of $\mathcal{H}$, then quasigeodesics are said to satisfy* **bounded penetration** *and X is said to be* **strongly hyperbolic** *relative to the collection $\mathcal{H}$.*

Lemma 1.2. *(See Lemma 4.5 and Proposition 4.6 of* [13] *and Theorem 5.3 of Klarreich* [18]*) Given δ, C, D there exists Δ such that if X is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex D-separated sets. then,*

(1) *Electric quasi-geodesics electrically track hyperbolic geodesics: Given $P > 0$, there exists $K > 0$ with the following property: Let β be any electric P-quasigeodesic from x to y, and let γ be the hyperbolic geodesic from x to y. Then $\beta \subset N^e_K(\gamma)$.*

(2) *γ lies in a* hyperbolic *K-neighborhood of $N_0(\beta)$, where $N_0(\beta)$ denotes the zero neighborhood of β in the* electric metric.

(3) *Hyperbolicity: X is Δ-hyperbolic.*

Now, let $\alpha = [a, b]$ be a hyperbolic geodesic in X and β be an electric P-quasigeodesic without backtracking joining a, b. Starting from the left of β, replace each maximal subsegment, (with end-points p, q, say) lying within some $H \in \mathcal{H}$ by a hyperbolic geodesic $[p, q]$. The resulting **connected** path β_q is called an *electro-ambient representative* in X.

Lemma 1.3. [28] *Let X be a hyperbolic metric space and $\mathcal{H}$ a collection of ϵ neighborhoods of mutually cobounded quasiconvex sets; then any electro-ambient quasigeodesic is a quasigeodesic in X.*

We have a slightly stronger version of the second statement of Lemma 1.2 with the notion of electro-ambient quasigeodesics in place.

Lemma 1.4. [28] *Given δ, C, D, P there exists C_3 such that the following holds: Let (X, d) be a δ-hyperbolic metric space and $\mathcal{H}$ a family of C-quasiconvex, D-separated collection of quasiconvex subsets. Let (X, d_e) denote the electric space obtained by electrocuting elements of $\mathcal{H}$. Then, if α, β_q denote respectively a hyperbolic geodesic and an electro-ambient P-quasigeodesic with the same end-points, then α lies in a (hyperbolic) C_3 neighborhood of β_q.*

1.2. **Cannon-Thurston Maps.** Let (X, d_X) be a hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i : Y \to X$ denote inclusion.

Definition 1.5. *Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** $\hat{i}$ *from* $\widehat{Y}$ *to* $\widehat{X}$ *is a continuous extension of i.*

Lemma 2.1 of [25] below gives a necessary and sufficient condition for the existence of Cannon-Thurston maps.

Lemma 1.6. [25] *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists iff the following condition is satisfied:*
Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $y_0 \in Y$ any geodesic segment in Γ_G joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

The following reduction Theorem of Klarreich [18] enables us to extend results for closed surface groups to 3-manifolds with incompressible Scott cores.

Theorem 1.7. Theorem C of [18] *Let X and Y be proper, geodesic Gromov-hyperbolic spaces, $\{H_\alpha\}$ a collection of closed, disjoint, path-connected uniformly quasiconvex and uniformly separated subsets of X, and $h : X \to Y$ a coarsely Lipschitz map such that for every $\{H_\alpha\}$, the restriction $h|_\alpha$ extends continuously to a continuous map $h : \partial H_\alpha \to \partial Y$. Suppose further that*
1) $(X \setminus \bigcup_\alpha H_\alpha)$ and $(Y, \setminus \bigcup_\alpha h(H_\alpha))$ are path connected
2) the collection $\{h(H_\alpha)\}$ is also closed, disjoint, path-connected uniformly quasiconvex and uniformly separated in Y
3) the induced map $h_E : \mathcal{E}(X, \mathcal{H}) \to \mathcal{E}(Y, h(\mathcal{H}))$ is a quasi-isometry
Then h extends continuously to a continuous map $\partial h : \partial X \to \partial Y$.

In earlier work [28] we showed:

Theorem 1.8. [28] *Let $\rho : \pi_1(S) \to PSL_2(C)$ be a discrete faithful representation of a surface group with or without punctures, and without accidental parabolics. Let $M = \mathbb{H}^3/\rho(\pi_1(S))$. Let i be an embedding of S in M that induces a homotopy equivalence. Then the embedding $\tilde{i} : \widetilde{S} \to \widetilde{M} = \mathbb{H}^3$ extends continuously to a map $\hat{i} : \mathbb{D}^2 \to \mathbb{D}^3$. Further, the limit set of $\rho(\pi_1(S))$ is locally connected.*

In [29] we further showed the following.

Theorem 1.9. *Suppose a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries and has no accidental parabolics. Then the inclusion $\tilde{i} : \widetilde{S} \to \mathbb{H}^3$ extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

Combining Theorems 1.8 and 1.9 with 1.7 we get the following rather general result for hyperbolic 3-manifolds with incompressible Scott cores. Let G be a freely indecomposable Kleinian group and let $M = \mathbb{H}^3/G$ We take for X in Theorem 1.7, a geometrically finite Kleinian group with quotient manifold N and fundamental group abstractly isomorphic to G. We identify the convex core of N with a submanifold of M. Let $\{H_\alpha\}$ be the collection of peripheral surface groups.

Theorem 1.10. *Suppose a freely indecomposable group G acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries and has no accidental parabolics. Let M be* *the quotient and N be its compact core. Then the inclusion $\tilde{i} : \widetilde{N} \to \widetilde{M}$ extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending laminations of an end , or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

1.3. **Split Geometry.** We shall briefly recall the construction of models of *split geometry* from [28]. We shall also need the construction of certain quasiconvex ladder-like sets $\mathcal{L}_\lambda$.

Topologically, a **split subsurface** S^s of a surface S is a (possibly disconnected, proper) subsurface with boundary such that
1) each component of S^s is an essential subsurface of S.
2) no component of S^s is an annulus.
3) $S - S^s$ consists of a non-empty family of non-homotopic essential annuli, none of which are homotopic into the boundary of S^s.

Geometrically, we assume that S is given some finite volume hyperbolic structure. A split subsurface S^s of S has bounded geometry, i.e.
1) each boundary component of S^s is of length ϵ_0, and is in fact a component of the boundary of $N_k(\gamma)$, where γ is a hyperbolic geodesic on S, and $N_k(\gamma)$ denotes its k-neighborhood.
2) For any closed geodesic β on S, either $\beta \subset S - S^s$, or, the length of any component of $\beta \cap (S - S^s)$ is greater than ϵ_0.

Definition-Theorem 1.11. (WEAK SPLIT GEOMETRY) *We constructed in* [28] *the following from the Minsky model for a simply degenerate end:*
1) A sequence of split surfaces S^s_i exiting the end of M, where M is marked with a homeomorphism to $S \times J$ (J is $\mathbb{R}$ or $[0,\infty)$ according as M is totally or simply degenerate). $S^s_i \subset S \times \{i\}$.
2) A collection of Margulis tubes $\mathcal{T}$.
3) For each complementary annulus of S^s_i with core σ, there is a Margulis tube T whose core is freely homotopic to σ and such that T intersects the level i. (What this roughly means is that there is a T that contains the complementary annulus.) We say that T splits S^s_i.
4) There exist constants $\epsilon_0 > 0, K_0 > 1$ such that for all i, either there exists a Margulis tube splitting both S^s_i and S^s_{i+1}, or else $S_i (= S^s_i)$ and $S_{i+1} (= S^s_{i+1})$ have injectivity radius bounded below by ϵ_0 and bound a **thick block** *B_i, where a thick block is defined to be a K_0 bi-Lipschitz homeomorphic image of $S \times I$.*
5) $T \cap S^s_i$ is either empty or consists of a pair of boundary components of S^s_i that are parallel in S_i.
6) There is a uniform upper bound $n = n(M)$ on the number of surfaces that T splits.
A model manifold satisfying conditions (1)-(6) above is said to have **weak split geometry**.

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *not necessarily connected* S^s. However, its upper and lower boundaries need not be $S^s \times 1$ and $S^s \times 0$. We only require that the upper and lower boundaries be *split subsurfaces* of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. Connected components of split blocks are called **split components**. We demand that there is a *non-empty* collection of Margulis tubes splitting a split block. However we *do not* require that the upper (or lower) horizontal boundary of a split component K be connected. This happens due to the presence of *hanging tubes*. See figure below, where the left split component has four hanging tubes and the right split component has two hanging tubes. The vertical space between the components is the place where a Margulis tube splits the split block into two split components.

Note that the whole manifold M is then the union of
a) Thick blocks (homeomorphic to $S \times I$)
b) Split blocks (homeomorphic to $S^s \times I$ for some split surfaces)
c) Margulis tubes.

The union of thick blocks and split blocks give rise to the complement (in $M = S \times J$) of a special collection of Margulis tubes. Each of these Margulis tubes splits a uniformly bounded number of split blocks and might end in a hanging tube.

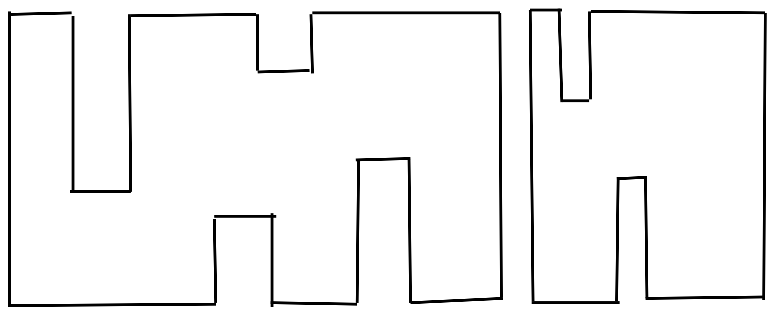

Figure 1: *Split Components of Split Block with hanging tubes*

We define a **welded split block** to be a split block with identifications as follows: Components of $\partial S^s \times 0$ are glued together if and only if they correspond to the same geodesic in $S - S^s$. The same is done for components of $\partial S^s \times 1$. A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also glue the boundary circles of their *lower or upper boundaries* by simply collapsing $S^1 \times [-\eta, \eta]$ to $S^1 \times \{0\}$. The same construction is repeated for all $i \geq 0$ by replacing $0, 1$ by $i, i+1$ respectively.

Let the metric product $S^1 \times [0, 1]$ be called the **standard annulus** if each horizontal S^1 has length ϵ_0. For hanging tubes the standard annulus will be taken to be $S^1 \times [0, 1/2]$.

Next, we require another pseudometric on B which we shall term the **tube-electrocuted metric**. We first define a map from each boundary annulus $S^1 \times I$ (or $S^1 \times [0, 1/2]$ for hanging annuli) to the corresponding standard annulus that is affine on the second factor and an isometry on the first. Now glue the mapping cylinder of this map to the boundary component. The resulting 'split block' has a number of standard annuli as its boundary components. Call the split block B^s with the above mapping cylinders attached, the *stabilized split block* B^{st}.

Glue boundary components of B^{st} corresponding to the same geodesic together to get the **tube electrocuted metric** on B as follows. Suppose that two boundary components of B^{st} correspond to the same geodesic γ. In this case, these boundary components are both of the form $S^1 \times I$ or $S^1 \times [0, \frac{1}{2}]$ where there is a projection onto the horizontal S^1 factor corresponding to γ. Let $S^1_l \times J$ and $S^1_r \times J$ denote these two boundary components (where J denotes I or $[0, \frac{1}{2}]$). Then each $S^1 \times \{x\}$ has length ϵ_0. Glue $S^1_l \times J$ to $S^1_r \times J$ by the natural 'identity map'. Finally, on each resulting $S^1 \times \{x\}$ put the zero metric. Thus the annulus $S^1 \times J$ obtained via this identification has the zero metric in the *horizontal direction* $S^1 \times \{x\}$ and the Euclidean metric in the *vertical direction* J. The resulting block will be called the **tube-electrocuted block** B_{tel} and the pseudometric on it will be denoted as d_{tel}. Note that B_{tel} is homeomorphic to $S \times I$. The operation of obtaining a *tube electrocuted block and metric* (B_{tel}, d_{tel}) from a split block B^s shall be called *tube electrocution*. Note that a tube electrocuted block is homeomorphic to $S \times I$.

A lift of a split component to the universal cover of the block $B = S \times I$ or $B_{tel} = S \times I$ shall be termed a **split component** $\widetilde{K}$ of $\widetilde{B}$ or $\widetilde{B_{tel}}$.

Also, let d_G be the (pseudo)-metric obtained by electrocuting the collection of split components $\widetilde{K}$ in $\widetilde{B_{tel}}$. d_G will be called the the **graph metric**.

Definition-Theorem 1.12. (SPLIT GEOMETRY) *In* [28] *we constructed a sequence of split surfaces that satisfy the following two conditions in addition to Conditions (1)-(6) of Definition-Theorem 1.11 for the Minsky model of a simply or totally degenerate surface group:*
7) Each split component $\widetilde{K} \subset \widetilde{B_i} \subset \widetilde{M}$ is (not necessarily uniformly) quasiconvex in the hyperbolic metric on $\widetilde{M}$.
8) Equip $\widetilde{M}$ with the graph-metric d_G obtained by electrocuting each split component $\widetilde{K}$. Then the convex hull $CH(\widetilde{K})$ of any split component $\widetilde{K}$ has uniformly bounded diameter in the metric d_G. We say that the components $\widetilde{K}$ are **uniformly graph-quasiconvex**. *It follows that $(\widetilde{M}, d_G)$ is a hyperbolic metric space.*
A model manifold satisfying conditions (1)-(8) above is said to have **split geometry**.
8A) To prove Property (8) we had shown the following in [28]*:*
There exists $C_0 \geq 1$ such that for every split component $K \subset M$ with bounding Margulis tube T, and for all $x \in K$ there exists a C_0-Lipschitz map of a hyperbolic surface passing through a C-neighborhood of x as well as a C-neighborhood of T. We also showed that any such surface has uniformly bounded diameter in the d_G metric.

From a geodesic $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \widetilde{B_0}$ we constructed in [27] a 'hyperbolic ladder' $\mathcal{L}_\lambda$ such that $\lambda_i = \mathcal{L}_\lambda \cap \tilde{S} \times \{i\} \subset \tilde{S}_i$ is an electro-ambient quasigeodesic in the (path) electric metric on $\tilde{S}_i$ induced by the graph metric d_G on $\tilde{M}$. λ_{i+1} is constructed inductively from λ_i (in [27]) by 'flowing λ_i up' in the block $\widetilde{B_i}$. More precisely, $\widetilde{B_i}$ has a natural product structure and is bounded by $\widetilde{S_i}$ and $\widetilde{S_{i+1}}$. Given λ_i joining $p_i, q_i \in \widetilde{S_i}$, there exist points $p_{i+1}, q_{i+1} \in \widetilde{S_{i+1}}$ lying vertically above p_i, q_i respectively. λ_{i+1} is the electro-ambient geodesic in $\widetilde{S_{i+1}}$ (equipped with the electric metric) joining p_{i+1}, q_{i+1}.

We also constructed a large-scale retract $\Pi_\lambda : \tilde{M} \to \mathcal{L}_\lambda$ such that the restriction π_i of Π_λ to $\tilde{S} \times \{i\}$ is, roughly speaking, a nearest-point retract of $\tilde{S} \times \{i\}$ onto λ_i in the (path) electric metric on $\tilde{S}_i$.

We have the following basic theorem from [28]

Theorem 1.13. [28] *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \widetilde{M} \to \mathcal{L}_\lambda$ satisfies:*

Then $d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y) + C$.

2. Free Groups and Finitely Generated Kleinian Groups

Let G be a free geometrically infinite Kleinian groups. Agol [1], and independently, Gabai and Calegari [15] have shown that $M = \mathbf{H}^3/G$ is topologically tame, and hence, by work of Canary [8], geometrically tame. Then M is homeomorphic to the interior of a handlebody with boundary S, say. Further, from the proof of the Ending Lamination Conjecture for such groups [6] [3], we know that a neighborhood E of the end of M is bi-Lipschitz homeomorphic to a Minsky model for a simply degenerate surface group, where the surface corresponds to S. The same conclusion goes through in the more general situation of an arbitrary finitely generated Kleinian group.

2.1. The Masur Domain.

Definition 2.1. *A homeomorphism $h : E \to S \times [0, \infty)$ is said to be type-preserving, if all and only the cusps of E are mapped to cusps of $S \times [0, \infty)$.*

Theorem 2.2. [1] [15] [3] [6] *Let G be a finitely generated Kleinian group and $M = \mathbf{H}^3/G$. Let C denote an augmented Scott core of M. Let E_1 be a geometrically infinite end of $M \setminus C$. Then E_1 is homeomorphic (via a type-preserving homeomorphism) to a topological product $S \times [0, \infty)$ for a hyperbolic surface S of finite area. Further, there exists a neighborhood E of the end corresponding to E_1 such that E is bi-Lipschitz homeomorphic to a Minsky model for $S \times [0, \infty)$.*

Some ambiguity remains in the statement of Theorem 2.2 above. This lies in the choice of the ending lamination for E used to build the Minsky model. Since $i : S \subset E$ is type-preserving, no parabolic element of S bounds a compressing disk.

As before, let $\mathcal{ML}(S)$ be the space of measured laminations. Let $\mathcal{D}(S)$ be the subset of $\mathcal{ML}(S)$ consisting of a weighted union of disjoint meridians (i.e. boundaries of compression disks lying on S). Let $cl(\mathcal{D}(S))$ denote the closure of $\mathcal{D}(S)$. Define the **Masur domain** of S by

$$\mathcal{MD}(S) = \{\lambda \in \mathcal{ML}(S) : i(\lambda, \mu) > 0 \text{ for all } \mu \in cl(\mathcal{D}(S))$$

Remark: Except for the case where S has only one isotopy class of compressing disks, the same definition of the **Masur Domain** as above goes through. Else, when S has only one isotopy class of compressing disks, we define

$$\mathcal{MD}(S) = \{\lambda \in \mathcal{ML}(S) : i(\lambda, \mu) > 0 \text{ for any } \mu \text{ that is disjoint from a compressing disk } \}$$

Now, let $M = H \cup_S E$, where H is a handlebody. Let $Mod_0(S)$ denote the subgroup of the mapping class group of S generated by Dehn twists along curves that bound embedded disks in H. As is customary, a prefix $\mathcal{P}$ will indicate projectization. It was shown by Otal [31] (see also McCarthy and Papadopoulos [19]) that $Mod_0(S)$ acts properly discontinuously on $\mathcal{PMD}(S)$ with limit set $Pcl(\mathcal{D}(S))$.

The ending lamination is well-defined upto the action of $Mod_0(S)$. (See Canary [8].)

Theorem 2.3. [8] *For any finitely generated Kleinian group, the ending lamination λ facing a surface S with a compressing disk lies in the Masur Domain.*

For our purposes we shall mostly be satisfied with the fact that E is bi-lipschitz homeomorphic to *some* Minsky model, and hence, by Definition-Theorem 1.12 of *split geometry.*

2.2. **Incompressibility of Split Components.** We would like to show that sufficiently deep within E, all split components are incompressible *in* M. Recall that splitting tubes corespond to *thin Margulis tubes* in the split geometry model built from the Minsky model. We state and prove the propositions for free groups without parabolics and tag on remarks to indicate the generalisations for arbitrary finitely generated Kleinian groups.

Proposition 2.4. *Let $M = H \cup E$. Suppose that E has split geometry with splitting Margulis tubes (if any) corresponding to curves in the hierarchy. Then there exists $E_2 \subset E$ such that*

(1) *E_2 is homeomorphic to $S \times [0, \infty)$ by a type-preserving homeomorphism and consists of a union of blocks from the split geometry model for E.*
(2) *All split components of E_2 are incompressible, i.e. if K is a split component of E_2, then the inclusion $i : K \to M$ induces an injective map $i_* : \pi_1(K) \to \pi_1(M)$.*

Proof: Suppose not, Then there exists a sequence of split components K_i exiting the end E such that $i_* : \pi_1(K) \to \pi_1(M)$ is not injective. Since K_i are split components, $K_i = S_i \times I$ for some subsurface S_i of S. By the Loop Theorem (See for instance, Hempel [16]), there exist simple closed curves $\sigma_i \subset S_i$ such that σ_i bound embedded topological disks in H. Hence $\sigma_i \in \mathcal{CD}(S)$. Let T_i be a splitting tube bounding K_i. Then T_i has a core curve α_i in the hierarchy. It follows that $d_{CC}(\alpha_i, \mathcal{D}(S)) = 1$. Since any such sequence of curves α_i converges to the ending lamination λ corresponding to the end E, it follows that $\lambda \in cl(\mathcal{D}(S))$ and hence cannot lie in the Masur domain. This contradicts Theorem 2.3. The proposition follows. $\Box$

Remark 2.5. *The above proposition and its proof go through verbatim for arbitrary finitely generated Kleinian groups, including those with parabolics.*

We shall henceforth (by slight abuse of notation) assume that $M = H \cup E$ and that E has split geometry with splitting tubes corresponding to hierarchy curves. Further, we shall assume that for any split block of E, any split component is incompressible in $\widetilde{M}$.

Note: The split geometry model for E induces one for E_2. However, this only means that a split geometry model for E has been constructed in the intrinsic metric on E. Proposition 2.4 allows us to translate this information into M and regard split blocks as blocks of M.

First, by the Thurston-Canary covering theorem [9], we have the following.

Lemma 2.6. *If K is a split component, then $\pi_1(K) \subset \pi_1(M)$ is geometrically finite (Schottky, in the absence of parabolics).*

Next, we construct a graph metric on M and $\widetilde{M}$ as as in the discussion preceding Definition-Theorem 1.12, with the modification that H is regarded as a single block, and retains its *intrinsic* metric. Then, in Definition-Theorem 1.12, Condition (8A) gives the following.

Proposition 2.7. *If K is a split component, then $\widetilde{K}$ is uniformly graph-quasiconvex in $\widetilde{M}$.*

Remark 2.8. *The above Proposition used the split geometry model as in Definition-Theorem 1.12 to conclude that for any split component K of the end E, $\widetilde{K}$ is uniformly graph-quasiconvex in $\widetilde{M}$. Nothing special about H being a handlebody was used, except for the definition of the Masur domain. The definition of a Masur domain for arbitrary freely decomposable Kleinian groups has been given earlier. With this definition, Proposition 2.7 and its proof go through in this generality, even in the presence of parabolics.*

Now, if $CH(\widetilde{K})$ denote the collection of convex hulls of split components $\widetilde{K}$ of $\widetilde{M}$, then by Lemma 1.2, $\widetilde{M}$ is hyperbolic relative to the collection $CH(\widetilde{K})$. Let d_2 be the electric metric on $\widetilde{M}$ obtained by electrocuting the collection $CH(\widetilde{K})$. Then $\widetilde{M}$ is hyperbolic with the pseudometric d_2.

Also, let d_G be the graph metric obtained by electrocuting the collection $\widetilde{K}$. Since the components $\widetilde{K}$ are uniformly graph-quasiconvex by Proposition 2.7 (i.e. each $CH(\widetilde{K})$ has uniformly bounded diameter in the metric d_G), it follows that the identity map from $(\widetilde{M}, d_2)$ to $(\widetilde{M}, d_G)$ is a quasi-isometry. Hence, we conclude:

Proposition 2.9. *$(\widetilde{M}, d_G)$ is a hyperbolic metric space.*

Remark 2.10. *As in Remark 2.8, the above proposition and its proof goes through for arbitrary finitely generated Kleinian groups without parabolics. For parabolics a bit more work involving partial electrocution (Section 8 of* [28]*) gives the same conclusion. We first partially electrocute $\mathbb{Z}$-cusps and then apply the above arguments. This shows that the partially electrocuted model with the d_G metric is hyperbolic.*

2.3. **Constructing Quasidisks.** The construction in this subsection may be regarded as a *graph-metrized coarse* analogue of an unpublished construction due to Miyachi [26] (See also Souto [32]). We choose a collection of g curves $\sigma_1 \cdots \sigma_g$ on S bounding disks $D_1 \cdots D_g$ with neighborhoods $D_i \times (-\epsilon, \epsilon)$ (assuming that the genus of S is g). Further, assume that $H \setminus \bigcup_i D_i \times (-\epsilon, \epsilon)$ is a ball. Also assume that σ_i are geodesics in the intrinsic metric on S. Next, let E be described as a union of contiguous blocks B_k, where each B_k is either a split block, or a thick block.

Further, let $\partial B_k = S_{k-1} \cup S_k$ with S_k the *upper boundary* and S_{k-1} the *lower boundary.* Also let $S = S_0$ and $\sigma_i = \sigma_{i0}$. Let σ_{ik} be the shortest closed curve in the split metric on S_k (i.e. in the pseudometric obtained by electrocuting annular intersections of splitting Margulis tubes with the split level surface S_k). Let $A_i = D_i \bigcup_k \sigma_{ik} \subset M$ be the union of the disk D_i and the quasi-annulus $\bigcup_k \sigma_{ik}$. Then any lift of A_i to $\widetilde{M}$ is isometric to A_i as D_i is homotopically trivial and σ_{ik} are all freely homotopic to $\sigma_i = \sigma_{i0} = \partial D_i$.

We want to show that A_i are quasiconvex in $(\widetilde{M}, d_G)$ which is hyperbolic by Proposition 2.9.

qi Rays

Fix a σ and the disk D it bounds. Let $A = D \bigcup_k \sigma_k \subset M$, where $\sigma_k \subset S_k$. Lift $\bigcup_k \sigma_k$ to $\widetilde{E}$ such that $\widetilde{\sigma_k} = \lambda_k \subset \widetilde{S_k}$. We then have the following from [28].

Lemma 2.11. [28] *There exists $C \geq 0$ such that for $x_k \in \lambda_k$ there exists $x_{k-1} \in \lambda_{k-1}$ with $d_G(x_k, x_{k-1}) \leq C$. Similarly there exists $x_{k+1} \in \lambda_{k+1}$ with $d_G(x_k, x_{k+1}) \leq C$. Also, for all k there exists $B(k)$ such that for all $x_k \in \lambda_k$ there exists $x_{k-1} \in \lambda_{k-1}$ with $d(x_k, x_{k-1}) \leq B(k)$. Hence, for all n and $x \in \lambda_n$, there exists a C-quasigeodesic ray r such that $r(k) \in \lambda_k \subset \mathcal{L}_\lambda$ for all k and $r(n) = x$.*

Further, by construction of split blocks, $d_G(x_i, S_{i-1}) = 1$. Therefore inductively, $d_G(x_i, S_j) = |i - j|$. Hence $d_G(x_i, x_j) \geq |i - j|$. By construction, $d_G(x_i, x_j) \leq C|i - j|$.

Hence, given $p \in \lambda_i$ the sequence of points $x_n, n \in \mathbb{N} \cup \{0\}$ with $x_i = p$ gives by Lemma 2.11 above, a quasigeodesic in the d_G-metric. Such quasigeodesics shall be referred to as *d_G-quasigeodesic rays*.

Ahter projecting $\widetilde{E}$ to $\widetilde{M} \setminus \widetilde{H}$ we have the following conclusion.

Lemma 2.12. *There exists $C \geq 0$ such that for all k there exists B_k such that for all $x_{ik} \in \sigma_{ik}$ there exists $x_{i,k-1} \in \sigma_{i,k-1}$ with $d_G(x_{ik}, x_{i,k-1}) \leq C$ and $d(x_{ik}, x_{i,k-1}) \leq B_k$.*

Further, by construction of split blocks, $d_G(x_{ik}, S_{k-1}) = 1$.

Hence, given $p \in \sigma_{ik}$ the sequence of points $p = x_{ik}, \cdots, x_{i0}$ gives by Lemma 2.12 above, a quasigeodesic in the d_G-metric lying entirely on A_i joining p to a point $q \in D_i$. We choose a point $z_i \in D_i$ (quite arbitrarily) and extend any quasigeodesic constructed as above by adding on a path from q to z_i lying entirely in D_i and having uniformly bounded length (This can be done easily as D_i has bounded diameter.)

Proposition 2.13. *There exists $C_0 \geq 0$ such that each A_j is C_0-quasiconvex in $(\widetilde{M}, d_G)$.*

Proof: By Lemma 2.12 and the discussion above, it follows that there exist $K \geq 1$ such that for any two points p_1, p_2 in A_j there exist K- quasi-geodesics γ_1, γ_2 to z_j. By Proposition 2.9 we also have that $(\widetilde{M}, d_G)$ is hyperbolic. Hence any geodesic α_i $(i = 1, 2)$ joining p_i to z_j lies in some K_1 neighborhood of γ_i. Further, by hyperbolicity of $(\widetilde{M}, d_G)$, we conclude that a geodesic β joining p_1, p_2 lies in a K_2-nieghborhood of $\alpha_1 \cup \alpha_2$. Hence, finally, β lies in a $(K_1 + K_2)$ neighborhood of $\gamma_1 \cup \gamma_2 \subset A_j$. Choosing $C_0 = K_1 + K_2$, we are through. $\Box$

Remark 2.14. *The above proposition and its proof also go through for arbitrary finitely generated Kleinian groups without parabolics. For finitely generated Kleinian groups with parabolics we modify the model by first partially electrocuting the $\mathbb{Z}$-cusps of the space (cf. Section 8 of* [28]*) and the proof of Proposition 2.13 goes through for any quasidisk A_k corresponding to a compressing disk.*

2.4. **Cannon-Thurston Maps for Free Groups.** We now want to show that if $\lambda = [a, b]$ is a geodesic in the intrinsic metric on $\widetilde{H}$ joining $a, b \in \widetilde{H}$, and lying outside a large ball about a fixed reference pint $p \in \widetilde{H} \subset \widetilde{M}$, then we want to show that the hyperbolic geodesic λ_h joining $a, b \in \widetilde{M}$ also lies outside a large ball about

p in $\widetilde{M}$. This would guarantee the existence of a Cannon-Thurston Map by Lemma 1.6.

Since each (lift of) D_i separates $\widetilde{H}$ and since λ lies outside a large ball about p in $\widetilde{H}$, we conclude that there exists such a lift D lying outside a large ball in $\widetilde{H}$ and that λ lies in the component of $\widetilde{H}$ not containing p.

The rest of the argument for the existence of Cannon-Thurston maps is in the spirit of [28] thorugh there are technical differences. We start off by constructing *admissible paths* in the (thick and split) building blocks of split geometry. From these we construct an *electro-ambient quasigeodesic* λ_q joining the end-points of λ. Finally, Lemma 1.4 assures us that the hyperbolic geodesic λ_h lies in a bounded neighborhood of λ_q. Hence, it is enough to show that λ_q lies outside a large ball about p.

Let λ_G be a geodesic in the graph metric joining a, b. We assume that λ_G is built up of A-*admissible paths*.

A-admissible paths are defined as follows:

Case 1: Thick Block

(1) Horizontal geodesic subarcs of σ_{ik}, $i = 1 \cdots g$.
(2) Vertical segments of length 1 joining $x \times \{0\}$ to $x \times \{1\}$ where at least one of $(x, 0)$ or $(x, 1)$ lies in some A_i.
(3) Horizontal geodesic segments lying in some (uniform) C-neighborhood of A_i, with at least one end-point on A_i.
(4) lifts of hyperbolic geodesics in some S_k.

Case 2: Split Block

Let $B = S \times [k, k+1]$ be a split block, where each (x, k) is connected by a geodesic segment of zero electric length and hyperbolic length $\leq C(B)$ (due to bounded thickness of B) to $(x, k+1)$.

By Proposition 2.12 there exists $C > 0$ (uniform constant) and $K = K(B)$ such that for all $(x, k) \in \sigma_{ik}$, $(x, k+1)$ lies in a (d_G) C-neighborhood and a hyperbolic K-neighborhood of $\sigma_{i,k+1}$. Similarly, for all $(x, k+1) \in \sigma_{i,k+1}$, (x, k) lies in a (d_G) C-neighborhood and a hyperbolic K-neighborhood of σ_{ik}. The **A_i-admissible paths** in the lift $\widetilde{M}$ consist of the following:

(1) Horizontal subarcs of σ_{ik}, $i = 1 \cdots j$.
(2) Vertical segments of electric length one and hyperbolic length $\leq l_B$ (the upper bound on thickness for the block B) joining $x \times \{k\}$ to $x \times \{k+1\}$, where at least one of (x, k) and $(x, k+1)$ lies on A_i.
(3) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(x, k+1)$ to a point on $\sigma_{i,k+1}$ for $(x, k) \in \sigma_{ik}$.
(4) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form (x, k) to a point on σ_{ik} for $(x, k+1) \in \sigma_{i,k+1}$.
(5) lifts of electro-ambient quasigeodesics in some S_k^s

Let us now return to λ_G which we have assumed to be built up of A-admissible paths. If λ_G meets some (lift of) A_i, then let p, q be the first and last points of intersection with A_i. Suppose $p \in \sigma_{im}$ and $q \in \sigma_{in}$. By Lemma 2.13, we can join p to q by an admissible quasigeodesic in the graph metric d_G on $\widetilde{M}$. By replacing the segment of λ_G betwen p and q by such paths for each such A_i, we have an

admissible quasigeodesic (which we refer to as λ_G by slight abuse of notation) having the following property:
Let $H(1)$ denote the component of $\widetilde{H} \setminus D_i$ that does not contain p. Let $\widetilde{S_0^s}(1)$ denote the boundary of $H(1)$. Let $\widetilde{S_k^s}(1)$ denote the component of $\widetilde{S_k^s} \setminus \sigma_{ik}$ parallel to $\widetilde{S_0^s}(1)$. Let W denote the union $H(1) \cup_k \widetilde{S_k^s}(1)$. We would like to say that $\lambda_G \subset W$. While this is not strictly, true, it is almost true in the following sense:
• *$\lambda_G \setminus W \cap B_k^s$ lies in a uniform C-neighborhood of W in the graph-metric d_G and a K_k-neighborhood (dependent on the block B_k^s) of W in the hyperbolic metric d.*
We shall refer below to W as the 'quasi-component' of $\widetilde{M} \setminus A_i$.

Recall further, that by Proposition 2.7, $(\widetilde{M}, d_G)$ is quasi-isometric to $(\widetilde{M}, d_2)$ (where d_2 is the metric obtained by electrocuting the quasiconvex sets $CH(\widetilde{K})$). Hence λ_G is a quasigeodesic in $(\widetilde{M}, d_2)$.

Definition 2.15. *A (k, ϵ) electro-ambient quasigeodesic γ in $\tilde{M}$ relative to the collection $CH(\tilde{K})$ is a (k, ϵ) quasigeodesic in $(\tilde{M}, d_2)$ such that in an ordering (from the left) of the sets $CH(\tilde{K})$ that γ meets, each $\gamma \cap CH(\tilde{K})$ is a geodesic in the hyperbolic metric on $CH(\tilde{K})$.*

We modify λ_G as in [28] below.
• **Step 1:** Assume without loss of generality that λ_G does not backtrack in $\widetilde{M}$ relative to the subsets $CH(\widetilde{K})$.
• **Step 2:** Next decompose λ_G into pieces β_i such that each β_i is

(1) either an admissible geodesic lying outside all $CH(\tilde{K})$'s; in this case, it is a (uniform) hyperbolic quasigeodesic
(2) or it is an admissible quasigeodesic lying fully inside some $CH(\tilde{K})$.

In order (from the left), modify the β_i's that lie wholly inside some $CH(\tilde{K})$ by replacing β_i with a hyperbolic geodesic α_i joining its end-points and lying wholly within $CH(\tilde{K})$. The resulting path α is

(1) a quasigeodesic in $(\widetilde{M}, d_2)$ with the same length as λ_G and having the same intersection pattern with the $CH(\tilde{K})$'s.
(2) If $\alpha_i \subset CH(\tilde{K})$, where $K \subset B_m^s$, then α_i lies in a C_m-(hyperbolic) neighborhood of β_i. Here C_m depends solely on the split block B_m^s in which K lies.

• **Step 3:** By Lemma 1.4, the hyperbolic geodesic λ_h joining the end-points a, b of λ lies in a (uniform) C' neighborhood of α.

We are now in a position to prove the Cannon-Thurston for Free Groups. To fix notation, let F be a free Kleinian group without parabolics. Let $M = \mathbf{H}^3/G$ and H a compact (Scott) core of M. $\widetilde{H}$ with its intrinsic metric is quasi-isometric to the Cayley graph Γ_F and so its intrinsic boundary may be idenitifed with the Cantor set ∂F thought of as the Gromov boundary of Γ_F. Let $\widehat{H}$ and $\widehat{M}$ denote the compactifications by adjoining ∂F and the limit set Λ_F to $\widetilde{H}$ and $\widetilde{M}$ respectively.

Theorem 2.16. Cannon-Thurston for Free Groups *The inclusion $i : \widetilde{H} \to \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{H} \to \widehat{M}$.*

Proof: Let p, λ, λ_G, β_i, α, λ_h be as above. By Lemma 1.6 it suffices to show that if λ lies outside a large ball about p in $\widetilde{H}$, then λ_h lies outside a large ball about p in $\widetilde{M}$. From the discussion preceding this theorem, it is enough to show that α lies outside a large ball about p (by Lemma 1.4 as λ_h lies in a neighborhood of α).

Now α consists of pieces α_i such that the following holds:
If $\alpha_i \subset CH(\tilde{K})$, where $K \subset B^s_m$, then α_i lies in a C_m-(hyperbolic) neighborhood of β_i. Here C_m depends solely on the split block B^s_m in which K lies.

Next, $\lambda_G \setminus W \cap B^s_m$ and hence β_i lies in a uniform C-neighborhood of W in the graph-metric d_G and a K_m-neighborhood (dependent on the block B^s_m) of W in the hyperbolic metric d. (Recall that W is the 'quasi-component' of $\widetilde{M} \setminus A_i$, where $A_i = D_i \cup_k \sigma_{ik}$ is a quasidisk coarsely separating $\widetilde{M}$ such that λ_G lies in the 'quasi-component' W not containing p.) Since λ lies outside a large ball about p in $\widetilde{H}$, we may assume that D_i lies outside a large ball $B_N(p)$ about p.

Now from Lemma 2.12, we find that there exist constants $c', k_0, \cdots k_m, \cdots$ so that for any $x \in \sigma_{im} \subset A_i$,

$$d(x,p) \geq max\{c'm, N - \Sigma_{i=0\cdots m} k_m\}$$

The first quantity follows from a (uniform) lower bound on thickness of blocks and the second from (possibly non-uniform) upper bounds.

The right hand side of the above expression attains a minimum $M(N)$ for $c'm = N - \Sigma_{i=0\cdots m} k_m$. Then $M(N) \to \infty$ as $N \to \infty$. Note that this means that W lies outside an $M(N)$ ball about p in $\widetilde{M}$ if λ lies outside $B_N(p)$ in $\widetilde{H}$.

We now repeat this argument for β_i's. $\lambda_G \setminus W \cap B^s_m$ lies in a K_m neighborhood of W. Hence as above,

$$d(\lambda_G, p) \geq max\{c'm, M(N) - K_m\}$$

Finally,

$$d(\alpha, p) \geq max\{c'm, M(N) - K_m - C_m\}$$

Again, the right hand side of the above expression attains a minimum $M'(N)$ for $c'm = M(N) - K_m - C_m$. Then $M'(N) \to \infty$ as $N \to \infty$. Since λ_h lies in a (uniformly) bounded neighborhood of α, it follows that λ_h lies outside a large ball about p in $\widetilde{M}$ if λ lies outside a large ball about p in $\widetilde{H}$. By Lemma 1.6 this implies the existence of Cannon-Thurston maps for the group F. $\Box$

2.5. **Finitely Generated Kleinian Groups.** In this subsection, we indicate the modifications necessary in the previous subsection to prove the analogous theorem for finitely generated Kleinian groups without parabolics.

Let G be a finitely generated Kleinian group without parabolics and G_f be a geometrically finite Kleinian group, abstractly isomorphic to G via a type-preserving isomorphism. Let H denote the convex core of $\mathbf{H}^3/G_f$ and let $M = \mathbf{H}^3/G$. Then there is a natural identification $i : H \to M$ of H with the Scott core of M. Let $\widetilde{i}$ indicate the lift of i to the universal cover.

First, suppose that H has incompresible boundary. Then Theorem 1.10 shows that a Cannon-Thurston map exists for $\widetilde{i} : \widetilde{H} \to \widetilde{M}$. The point pre-image description is also furnished by Theorem 1.10.

Else H may be decomposed as the disk-connected sum of H_1, $H_2 \cdots H_{m+1}$ for manifolds H_i where at most one of the H_i's is a handlebody without any parabolics (taken to be H_1 without loss of generality) and the rest are manifolds with incompressible boundary. Let $D_i, i = 1 \cdots m$ denote the separating compressing disks giving rise to the above decomposition. If one of the components is a handlebody we adjoin non-separating disks of the handlebody cutting it up into a ball as in the previous subsection. Since $m \geq 1$, there is at least one compressing disk.

Theorem 2.17. Cannon-Thurston for Kleinian Groups *The inclusion* $\widetilde{i}$: $\widetilde{H} \to \widetilde{M}$ *extends continuously to a map* $\hat{i} : \widehat{H} \to \widehat{M}$.

Proof: From Remarks 2.5, 2.10 and 2.14, we can construct quasidisks A_i corresponding to D_i as before and lift them to $\widetilde{M}$.

Now, let λ be a geodesic segment in $\widetilde{H}$ lying outside a large ball $B_N(p)$ for a fixed reference point p. λ may be decomposed into (at most) three pieces λ_-, λ_0 and λ_+. We choose this decomposition such that

(1) The middle piece λ_0 does not intersect any of the (lifts of the) compressing disks D_i in its interior. (We thus allow for the cases where λ_- and/or λ_+ are empty.)
(2) the common end-point (if any) $\lambda_- \cap \lambda_0$ lies on some D_i. The same is demanded of $\lambda_0 \cap \lambda_+$.
(3) the point q on λ nearest to p lies on λ_0

Two cases arise.

Case A: For a sequence of λ's lying outside larger and larger balls $B_N(p)$ about p, the last disk D_i that $[p, q]$ intersects lies outside large balls $B_M(p)$ where $M \to \infty$ as $N \to \infty$. This is exactly the case as in the proof of Theorem 2.17. The same proof goes through.

Case B: There is (upto subsequence) a fixed disk D_i that is the last disk that $[p, q]$ intersects. Since D_i is of uniformly bounded diameter, we may shift our base point to a point p' in the component $\widetilde{H_i}$ which is the lift of H_i 'on the other side of' D_i, i.e. having D_i on its boundary but not containing p. In this case, there exists a fixed N_0 such that λ lies outside $B_{(N-N_0)}(p')$. By shifting origin, we rewrite p' as p and $(N - N_0)$ as N.

Step 1: Now, $\lambda_0 \widetilde{H_i} \subset \widetilde{H}$ as it does not meet any disk D_i in its interior. Since H_i is either a handlebody without parabolics, or a manifold with incompressible boundary, then by Theorem 2.16 or Theorem 1.10
respectively, the hyperbolic geodesic λ_{0h} joining the end-points of λ_0 in $\widetilde{M}$ lies outside a large ball about p. Thus, there exists $M_1(N) \to \infty$ as $N \to \infty$ such that λ_{0h} lies outside a ball of radius $M_1(N)$ about p in $\widetilde{M}$.

Step 2: If λ_+ (or λ_-) is non-empty, then λ_- (or λ_+) is separated from p by a disk $D_i \subset \widetilde{H}$ lying outside $B_N(p)$. Then the quasidisk A_i is quasiconvex and lies outside a large ball of radius $M_2(N)$ about p. Again, by constructing admissible paths and electro-ambient quasigeodesics as in the proof of Theorem 2.16, we obtain a new function $M_3(N)$ such that $M_3(N) \to \infty$ as $N \to \infty$ and so that the hyperbolic geodesics λ_{-h} or λ_{+h} lie outside a ball of radius $M_3(N)$ about p.

Step 3: Therefore $\lambda_{-h} \cup \lambda_{0h} \cup \lambda_{+h}$ lies outside a ball of radius $M_4(N) = min\{M_1(N), M_3(N)\}$. Finally, since $\widetilde{M}$ is hyperbolic, the hyperbolic geodesic λ_h joining the end-points of λ lies outside a ball of radius $M(N) = M_4(N) - 2\delta$ about p. Also, $M(N) \to \infty$ as $N \to \infty$. Therefore, by Lemma 1.6, it follows that the inclusion $\widetilde{i} : \widetilde{H} \to \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{H} \to \widehat{M}$. This concludes the proof. $\square$

Remark: Suppose G has parabolics. Then the above proof almost goes through. The one crucial ingredient that is missing is an analogue of Theorem 1.7 for pared manifolds with incompressible boundary. (Theorem 1.7 is the reduction Theorem that allows results for surface groups to be pushed through for 3-manifolds with incompressible boundary.) Modulo such a reduction Theorem, the above proof goes through verbatim.

For manifolds of bounded geometry, a direct proof (without a reduction Theorem) of the Cannon-Thurston property for pared manifolds with incompressible boundary was given by the author in [30]. In the general situation a direct proof without using a reduction Theorem was also given for manifolds of split geometry in [27]. However this proof is rather cumbersome and in [12] we shall provide the necessary reduction Theorem.

Now, let $\widehat{G_F}$ denote the Floyd compactification of a group G (See [14]). McMullen conjectured in [21] that there exists a continuous extension of $i : \Gamma_G \to \widetilde{M}$ to a map from $\widehat{G_F}$ to $\widehat{M}$. It was shown by Floyd in [14] that there is a continuous map from $\widehat{G_F}$ to $\widehat{H}$. Combining this with an analogue of Theorem 2.17 above for Kleinian groups with parabolics, we would get a proof of the following conjecture of McMullen [21].

Conjecture 2.18. *For any finitely generated Kleinian group G and $M = \mathbf{H}^3/G$, there is a continuous extension $\hat{i} : \widehat{G_F} \to \widehat{M}$.*

3. Point pre-images of the Cannon-Thurston Map

Finally we determine the pre-images of points under the Cannon-Thurston map for degenerate free Kleinian groups G. We set up some notation for the purposes of this section. Let G be a free degenerate Kleinian group without parabolics. Suppose that G is not geometrically finite. Let $M = \mathbb{H}^3/G$ be the quotient manifold. Note that the limit set of G is all of the sphere at infinity. Hence M is its own convex core. Let H be a compact core of M. H is a handlebody whose inclusion into M induces a homotopy equivalence. In fact, M deformation retracts onto H. Then $\widetilde{H}$ is embedded in $\widetilde{M} = \mathbb{H}^3$. Let Γ denote the Cayley graph of G with respect to some finite generating set of G. Assume that Γ is embedded in $\widetilde{H}$. Let S denote the boundary surface of H. We assume that the ending lamination Λ_{EL} is a geodesic lamination on S equipped with some (any) hyperbolic metric. This is well-defined up to Dehn twists along simple closed curves in S that bound disks in H. To avoid this ambiguity we will refer to the ending lamination in the Masur domain as Λ_{ELH}. $M \setminus Int(H)$ is homeomorphic to $S \times [0, \infty)$ and is bi-lipschitz homeomorphic to an end M_S of a simply degenerate hyperbolic manifold without accidental parabolics [4] [6]. Thus $S \times [0, \infty) \subset M$ equipped with its *intrinsic* path metric is bi-lipschitz homeomorphic to M_S. We shall have need to pass interchangeably between these two below.

3.1. **EL leaves are CT leaves.** Let $i : \widetilde{H} \to \widetilde{M}$ denote the inclusion. Let ∂i denote the continuous extension of i to the boundary in Theorem 2.16. Note that the inclusion of Γ into $\widetilde{H}$ with its intrinsic metric is a quasi-isometry. So we might as well replace the inclusion of Γ into $\widetilde{M}$ by that of $\widetilde{H}$ into $\widetilde{M}$. We shall show that point pre-images under ∂i correspond to end-points of leaves of an ending lamination in the Masur domain.

The inclusion of S into H as its boundary induces a surjection of fundamental groups with infinitely generated kernel N. Let S_N denote the cover of S corresponding to N. Then $S_N \subset \widetilde{H} \subset \widetilde{M}$.

To distinguish between the ending lamination Λ_{ELH} (in the Masur domain) and bi-infinite geodesics whose end-points are identified by ∂i, we make the following definition.

Definition 3.1. *A* **CT leaf** λ_{CT} *is a bi-infinite geodesic whose end-points are identified by* ∂i.
An **EL leaf** λ_{EL} *is a bi-infinite geodesic whose end-points are ideal boundary points of either a leaf of the ending lamination, or a complementary ideal polygon.*

We shall show that
- **An** *EL leaf* **is an** *CT leaf***.**
- **A** *CT leaf* **is an** *EL leaf***.**

Proposition 3.2. EL is CT *Let G be a free degenerate Kleinian group without parabolics. Let a, b be either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon. Then $\partial i(a) = \partial i(b)$*

Proof: This is almost identical to Proposition 2.1 of [29]. However, since the setup is somewhat different we include a proof. Take a sequence of short geodesics σ_i exiting the end. Let α_i be geodesics in the intrinsic metric on the boundary S (of H) freely homotopic to σ_i. By topological tameness [1] [15] and geometric tameness ([35] Ch. 9) we may assume further that α_i's are simple closed curves on S. Join α_i to σ_i by the shortest geodesic τ_i in $S \times [0, \infty)$ connecting the two curves. Then the collection α_i converges to the ending lamination on S ([35] Ch. 9). Also, in $S_N \subset \widetilde{H} \subset \widetilde{M}$, we obtain segments $\alpha_{fi} \subset \widetilde{S}$ which are finite segments whose end-points are identified by the covering map $P : \widetilde{S \times [0, \infty)} \to S \times [0, \infty)$. We also assume that P is injective restricted to the interior of α_{fi}'s mapping to α_i. Similarly there exist segments $\sigma_{fi} \subset S_N$ which are finite segments whose end-points are identified by the covering map $P : \widetilde{M} \to M$. We also assume that P is injective restricted to the interior of α_{fi}'s. The finite segments σ_{fi} and α_{fi} are chosen in such a way that there exist lifts τ_{1i}, τ_{2i}, joining end-points of α_{fi} to corresponding end-points of σ_{fi}. The union of these four pieces looks like a trapezium (see below).

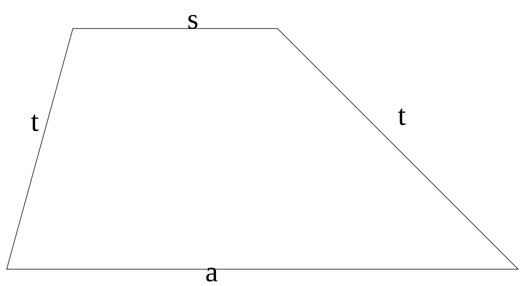

Figure: *Trapezium*

Next, given any leaf λ of the ending lamination, we may choose translates of the finite segments α_{fi} (under the action of $\pi_1(H)$) appropriately, such that they converge to λ in S_N. For each α_{fi}, let

$$\beta_{fi} = \tau_{1i} \circ \sigma_{fi} \circ \overline{\tau_{2i}}$$

where $\overline{\tau_{2i}}$ denotes τ_{2i} with orientation reversed. Then β_{fi}'s are uniform hyperbolic quasigeodesics in $\widetilde{M}$. If the translates of α_{fi} we are considering have end-points lying outside large balls around a fixed reference point $p \in S_N$, it is easy to check that β_{fi}'s lie outside large balls about p in $\widetilde{M}$.

At this stage we invoke the existence theorem for Cannon-Thurston maps, Theorem 2.16. Since α_{fi}'s converge to λ and there exist uniform hyperbolic quasigeodesics β_{fi}, joining the end-points of α_{fi} and exiting all compact sets, it follows that $\partial i(a) = \partial i(b)$, where a, b denote the boundary points of λ. The Proposition follows. $\Box$

Any finitely generated Kleinian group is geometrically tame ([1] [15] [35] Ch. 9) and has finitely many ends. Observe that the proof of the above Proposition used the freeness of G only at the stage of applying Theorem 2.16. The same proof goes through verbatim for freely decomposable Kleinian groups with degenerate ends. The only modification to the above proof is that we consider one end of the manifold M at a time (and the pigeon-hole principle) along with Theorem 2.17 in place of Theorem 2.16 to obtain the following Proposition.

Proposition 3.3. EL is CT - General Case *Let G be a freely decomposable Kleinian group without parabolics. Let a, b be either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon. Then $\partial i(a) = \partial i(b)$*

3.2. **CT leaves are EL leaves.** We restate Theorem 1.9 in a form that we shall use. Recall that $M \setminus Int(H)$ is homeomorphic to $S \times [0, \infty)$ and is bi-lipschitz homeomorphic to an end M_S of a simply degenerate hyperbolic manifold without accidental parabolics [4] [6]. Hence by Theorem 1.9 we have the following.

Theorem 3.4. [29] *Let S, M_S be as above. Then the inclusion $\tilde{j} : \widetilde{S} \to \widetilde{M_S}$ extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination Λ_{ELS} of M_S, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

Recall that $\Gamma \subset \widetilde{H} \subset \widetilde{M}$. By Theorem 2.16 the inclusion $i : \Gamma \to \widetilde{M}$ extends continuously to a map between the Gromov compactifications $\hat{i} : \widehat{\Gamma} \to \mathbb{D}^3$. Let ∂i denote the values of the above continuous extension to the boundary. Suppose $\partial i(a) = \partial i(b)$. Λ_{EL} is the ending lamination of M regarded as a subset of S. Let Λ_{ELG} denote Λ_{EL} lifted to $S_G = \partial \widetilde{H}$, which is a cover of S. We want to show that a, b are the end-points of a leaf of Λ_{ELG}. Suppose $(a, b)_\Gamma$ is the bi-infinite geodesic from a to b in $\Gamma \subset \widetilde{M}$. Assume without loss of generality that (a, b) passes through $1 \in \Gamma$. Let $a_k \to a$ and $b_k \to b$. Let $\overline{a_k b_k}$ denote the geodesic in $\widetilde{M}$ joining a_k, b_k. By continuity of the Cannon-Thurston map (Theorem 2.16) there exists $N(k) \to \infty$ as $k \to \infty$ such that $\overline{a_k b_k}$ lies outside an $N(k)$ ball about $1 \in \Gamma \subset \widetilde{M}$, where radius is measured in the hyperbolic metric on $\widetilde{M}$. Isotoping $\overline{a_k b_k}$ slightly, we can assume without loss of generality that it meets $\Gamma \subset \widetilde{M}$ only at its end-points (since Γ is one dimensional). We can further isotop $\overline{a_k b_k}$ by a bounded amount (depending on the Hausdorff distance between $\widetilde{H}$ and $\Gamma \subset \widetilde{H}$) such that

1) there exist $c_k, d_k \in S_G = \partial \widetilde{H}$ with $d(a_k, c_k)$ and $d(b_k, d_k)$ are uniformly bounded (independent of k)

2) if $\overline{c_kd_k}$ denotes the subpath of $\overline{a_kb_k}$ between c_k, d_k then (modifying $N(k)$ by an additive constant if necessary) $\overline{c_kd_k}$ lies outside an $N(k)$ ball about $1 \in \Gamma \subset \widetilde{M}$.
3) $\overline{c_kd_k}$ intersects $\widetilde{H}$ only at the endpoints c_k, d_k.

Let $[c_k, d_k]_{S_G}$ denote the geodesic in the intrinsic metric on S_G which is homotopic (rel. endpoints) to $\overline{c_kd_k}$ in $\widetilde{M} \setminus Int(\widetilde{H})$. Since G is free, we can assume that its Cayley graph is a tree and (since $\widetilde{H}$ is quasi-isometric to Γ) $[c_k, d_k]_{S_G}$ passes through a point $o_k \in S_G$ at a uniformly bounded neighborhood of 1. Lift $[c_k, d_k]_{S_G}$ to some geodesic $[c_k, d_k] \subset \widetilde{S} \subset \widetilde{M_S}$ in the intrinsic metric on $\widetilde{S}$. Further assume that there exists some fixed $o \in \widetilde{S}$ such that the corresponding lift o_k' of o_k lies in a uniformly bounded neighborhood of o. Let $\overline{c_kd_k}_{S_G}$ denote the corresponding lift of $\overline{c_kd_k}$ having the same endpoints as $[c_k, d_k]_{S_G}$ (such a choice is possible as $[c_k, d_k]_{S_G}$ and $\overline{c_kd_k}$ are homotopic rel. endpoints in the complement of $Int(\widetilde{H})$ in $\widetilde{M}$). It follows that $\overline{c_kd_k}_{S_G}$ lies outside an $N(k)$-ball about o_k' in $\widetilde{M_S}$. Hence (modifying $N(k)$ by a further additive constant if necessary), $\overline{c_kd_k}_{S_G}$ lies outside an $N(k)$-ball about $o \in \widetilde{M_S}$. Therefore, by the existence of Cannon-Thurston maps for $j : \widetilde{S} \to \widetilde{M_S}$ (Theorem 3.4) it follows that if $[c_\infty d_\infty]_{S_G}$ denotes any subsequential limit of the segments $[c_k, d_k]_{S_G}$ on $\widetilde{S}$, then $\partial j(c_\infty) = \partial j(d_\infty)$ and hence again by Theorem 3.4 c_∞, d_∞ are end-points of leaves of the ending lamination Λ_{ELS} of $S \subset M_S$. Finally, since (c_∞, d_∞) are bi-infinite geodesics passing through a bounded neighborhood of o, they project to leaves of Λ_{ELG} in S_G. These leaves are also well-defined as leaves of the ending lamination Λ_{ELH} as leaves of the ending lamination of M regarded as an element of the Masur domain. We have thus finally shown that $\Lambda_{CT} \subset \Lambda_{ELH}$. Combining this with Proposition 3.2 and Theorem 2.16 we have the main Theorem of this paper.

Theorem 3.5. *Let G be a free degenerate Kleinian group without parabolics. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$. Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma$ of Γ. Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial\Gamma$ if and only if a, b are either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon.*

Note that in the proof of Theorem 3.5 we have used freeness of G to conclude only two things:
1) The manifold M has exactly one end.
2) The path λ in $\widetilde{H}$ can be isotoped off a tree representing the Cayley graph of G embedded in $\widetilde{H}$.

To prove an analogue of Theorem 3.5 for arbitrary finitely generated Kleinian groups without parabolics we continue with the notation that M is a hyperbolic manifold with compact core H. Then M has finitely many ends. We first note, that if $\lambda = (a_\infty, b_\infty)$ is a CT leaf then there exist $a_n \to a_\infty$ and $b_n \to b_\infty$ such that the geodesic realizations μ_n of $[a_n, b_n]$ in $\widetilde{M}$ leave arbitrarily large compact sets. Since we may assume that $\widetilde{M} \setminus \widetilde{H}$ consists of lifts of the ends of M to $\widetilde{M}$. If μ_n intersects more than one such lift, it follows that there will be a subsegment ν_n of μ_n such that
1) ν_n is contained entirely in one of these lifts of the ends

2) Endpoints c_n, d_n of ν_n lie on $\widetilde{H}$
3) $c_n \to a_\infty$ and $d_n \to b_\infty$
We may therefore assume without loss of generality that μ_n lies in precisely one of the lifts of the ends E of M. If $S = H \cap E$ be its boundary then the ending lamination lies in the boundary of the hyperbolic group $j_*(\pi_1(S))$, where $j : S \to M$ is inclusion.

Fact (2) now goes through for arbitrary finitely generated Kleinian groups without parabolics, as the inclusion of the compact core into M is a homotopy equivalence and we are only interested in leaves which are limits of segments whose geodesic realizations lie inside the lift of a fixed end.

With this modification, and with Theorem 2.17 in place, the proof of Theorem 3.5 goes through for arbitrary finitely generated Kleinian groups without parabolics.

Theorem 3.6. *Let G be a finitely generated Kleinian group without parabolics. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$. Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma$ of Γ. Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial\Gamma$ if and only if a, b are either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon.*

4. Applications and Extensions

In this scetion we shall first describe extensions of the main Theorems of this paper to arbitrary finitely generated Kleinian groups. Next we shall mention an application to "Primitive Stable Representations" of free groups in $PSl_2(\mathbb{C})$. Finally we indicate an extension of the Sullivan-McMullen dictionary between complex dynamis and Kleinian groups.

4.1. **Parabolics.**

We have not dealt with the situation where G is allowed to have parabolics in this paper. As remarked after Theorem 2.17, the *proof* of Theorem 2.17 (Existence of Cannon-Thurston maps) almost goes through. The one crucial ingredient that is missing is an analogue of Theorem 1.7 for pared manifolds with incompressible boundary. Theorem 1.7 is the reduction Theorem that allows Theorems 1.8 and 1.9 for surface groups to be pushed through for 3-manifolds with incompressible boundary. Modulo such a reduction Theorem, we would have a complete proof of the Cannon-Thurston property for all finitely generated Kleinian groups, i.e. we would be able to remove the hypothesis of "no parabolics".

For manifolds of bounded geometry (with parabolics), a direct proof (without a reduction Theorem) of the Cannon-Thurston property for pared manifolds with incompressible boundary was given by the author in [30]. In the general situation a direct proof without using a reduction Theorem was also given for manifolds of split geometry in [27]. Since all 3-manifolds do have split geometry, this completes the proof of the Cannon-Thurston property for all finitely generated Kleinian groups. However, a) the proof in [27] is rather cumbersome.
b) it seems unlikely that the relevant portion of [27] will be published in this form.
c) Parts of the proof of this Theorem are only sketched in [27].
and in [12] we shall provide the necessary reduction Theorem.
d) The description of point-preimages under the Cannon-Thurston map have not been described for surfaces with punctures in [27].

In [12] we shall prove
a) A reduction Theorem along the lines of Klarreich's Theorem [18] for relatively hyperbolic spaces.
b) Describe point-preimages of Cannon-Thurston maps for punctured surface groups in terms of ending laminations. The bounded geometry case has been resolved by Bowditch in [5]. We will combine some of his techniques with some of ours in [29] to deduce this.

This will complete the programme of proving the existence of Cannon-Thurston maps for arbitrary finitely generated Kleinian groups and describing their structure in terms of point pre-images.

4.2. **Primitive Stable Representations.** In [24] Minsky introduced and studied primitive stable representations, an open set of $PSl_2(\mathbb{C})$ characters of a nonabelian free group, on which the action of the outer automorphism group is properly discontinuous, and which is strictly larger than the set of discrete, faithful convex-cocompact (i.e. Schottky) characters.

In [24] Minsky also conjectured that
Every discrete faithful representation of F without parabolics is primitive-stable.

Using the structure of the Cannon-Thurston map for handlebody groups, Jeon Woojin [36] has made considerable progress towards this conjecture. The connection between the Cannon-Thurston map for handlebody groups and primitive stable representations was indicated to the author by Yair Minsky in a personal communication.

4.3. **Extending the Sullivan-McMullen Dictionary.** A celebrated theorem of Yoccoz in Complex Dynamics (see Hubbard [17], or Milnor [22]) proves the local connectivity of certain Julia sets using a technique called 'puzzle pieces'. We shall not describe this in any detail. What we shall simply say is that it consists of a decomposition of a complex domain into pieces each of which under iteration by a quadratic map converges to a single point. The dynamical system can then be regarded as a semigroup $\mathbb{Z}_+$ of transformations acting on a complex domain.

Split components can be regarded as a 3-dimensional analogue of puzzle pieces. Let us try to justify this analogy. Suppose there is a group G acting on the manifold $\widetilde{M}$. Let $H \subset G$ denote the fundamental group of a split component. Let G/H denote the coset space. Then what we require first is that if one takes a sequence of elements g_i going to infinity in the coset space, the iterates of the split component converge to a point in the limit sphere. However, this does not give all the information as G does not act co-compactly on $\widetilde{M}$. In the cases we are interested in G/H correspond to normal directions to the split component lying within the block containing the split component. This does not help. To compensate, we look at the graph model. Here, there is no group in sight. However, normal directions can be salvaged from the **graph metric**. Thus, instead of going to infinity by iteration, we go to infinity in the graph metric. Further, the analogue of the requirement that iterates go to infinity, is that the visual diameter goes to zero as we move to infinity in the graph metric. This is easily ensured by hyperbolic quasiconvexity, and also follows easily from **graph quasiconvexity**. Note that **graph quasiconvexity** is a statement that gives uniform shrinking of visual diameter to zero as one goes to infinity.

Thus we extend the Sullivan-McMullen dictionary (see [34], [20])between Kleinian groups and complex dynamics by suggesting the following analogy:

(1) *Puzzle pieces* are analogous to **split components**

(2) *Convergence to a point under iteration* is analogous to **graph quasiconvexity**

One issue that gets clarified by the above analogy is a point raised by McMullen in [21]. McMullen indicates that though the Julia set $J(P_\theta)$, where

$$P_\theta(z) = e^{2\pi i\theta}z + z^2$$

need not be locally connected in general by a result of Sullivan [33], the limit set of the punctured torus groups are nevertheless locally connected. By extending the analogy of puzzle pieces, this issue is to an extent clarified.

An analogue of the $\mathbb{Z}_+$ dynamical system may also be extracted from the split geometry model. Note that each block corresponds to a splitting of the surface group, and hence an action on a tree. As $i \to \infty$, the split blocks B_i^s and hence the induced splittings also go to infinity, converging to a **free action of the surface group on an $\mathbb{R}$-tree dual to the ending lamination**. Thus iteration of the quadratic function correponds to taking a sequence of splittings of the surface group converging to a (particular) action on an $\mathbb{R}$-tree.

Problem: The building of the Minsky model and its bi-Lipschitz equivalence to a hyperbolic manifold [23] [7] gives rise to a speculation that there should be a purely combinatorial way of doing much of the work. Bowditch's rendering [3], [4] of the Minsky, Brock-Canary-Minsky results is a step in this direction. This paper brings out the possibility that the whole thing should be do-able purely in terms of actions on trees. Of course there is an action of the surface group on a tree dual to a pants decomposition. So we do have a starting point. However, one ought to be able to give a purely combinatorial description, *ab initio*, in terms of a sequence of actions of surface groups on trees converging to an action on an $\mathbb{R}$-tree. This would open up the possibility of extending these results (including those of this paper) to other hyperbolic groups with infinite automorphism groups, notably free groups.

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