

# Ending Laminations and Cannon-Thurston Maps

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## Abstract

In earlier work, we had shown that Cannon-Thurston maps exist for Kleinian surface groups. In this paper we prove that pre-images of points are precisely end-points of leaves of the ending lamination whenever the Cannon-Thurston map is not one-to-one. In particular, the Cannon-Thurston map is finite-to-one. This completes the proof of the conjectural picture of Cannon-Thurston maps. In conjunction with the Ending Lamination Conjecture of Thurston (now Theorem of Minsky, Masur-Minsky, Brock-Canary-Minsky), the main result of this paper gives a complete description of the isometry type of a simply degenerate hyperbolic manifold homeomorphic to  $S \times [0, \infty)$  in terms of the action of its fundamental group on the limit set and the domain of discontinuity.

*AMS Subject Classification: 57M50*

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# 1 Introduction

## 1.1 History and Statement of Results

The Ending Lamination Conjecture of Thurston (*ELC*) [28] (now a Theorem due to Minsky, Masur-Minsky, Brock-Canary-Minsky) may be regarded as a version of Mostow Rigidity for infinite volume hyperbolic 3-manifolds in general and those homeomorphic to  $S \times [0, \infty)$  in particular. We state this informally in the case that will interest us most in this paper, viz. simply degenerate manifolds homeomorphic to  $S \times [0, \infty)$ , where  $S$  is a closed surface.

**Theorem (Minsky, Masur-Minsky, Brock-Canary-Minsky) [18] [6] [14] [15]**

*Simply degenerate manifolds homeomorphic to  $S \times [0, \infty)$  are determined uniquely by two pieces of data:*

- 1) *The conformal structure on the boundary  $S \times \{0\}$*
- 2) *the ending lamination for the end.*

Thus purely topological data (the Ending Lamination) and asymptotic conformal data (conformal structure on the domain of discontinuity) determine upto isometry a simply degenerate hyperbolic manifold homeomorphic to  $S \times [0, \infty)$ . This is the content of the above foundational theorem of Minsky, Masur-Minsky, Brock-Canary-Minsky.

To complete the picture one would like to say that a simply degenerate hyperbolic manifold homeomorphic to  $S \times [0, \infty)$  is determined by "*purely asymptotic information*". More formally we would like to say that a simply degenerate hyperbolic manifold homeomorphic to  $S \times [0, \infty)$  is determined

by the two following pieces of data:

- 1) The conformal structure on the boundary  $S \times \{0\}$
- 2) the action of the surface group on its limit set.

The connection between the Ending Lamination Conjecture and this last characterisation is forged by the following conjecture (going back to Cannon and Thurston [7]).

**Conjecture 1.1** (*Cannon-Thurston [7]*)

*Suppose that a closed surface group  $\pi_1(S)$  acts freely and properly discontinuously on  $\mathbb{H}^3$  by isometries.*

1. *Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  extends continuously to a map  $\hat{i} : \hat{S} \rightarrow \hat{\mathbb{H}}^3$  between the compactifications  $\hat{S}$  and  $\hat{\mathbb{H}}^3$ .*
2. *Let  $\partial i$  denote the restriction of  $\hat{i}$  to the boundary and suppose that  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$  be two distinct points that are identified by such a Cannon-Thurston map (assuming that it exists) corresponding to a simply degenerate surface group (without accidental parabolics). Then  $a, b$  are either ideal end-points of a leaf of the ending lamination  $\mathcal{L}$ , or ideal boundary points of a complementary ideal polygon.*

Note that what the above conjecture states may be paraphrased in terms of limiting actions of  $\pi_1(S)$  on  $\mathbb{R}$ -trees. Now there is an action of  $\pi_1(S)$  on an  $\mathbb{R}$ -tree  $T$  dual to  $\mathcal{L}$  (See Shalen [26] for instance). Let  $\bar{T}$  denote the (Gromov) compactification of  $T$ . Then it follows from the above conjecture that the limit set is equivariantly (under the action of  $\pi_1(S)$ ) homeomorphic to  $\bar{T}$ .

Let explicitly describe how a proof of this conjecture would lead to the conjecture we started with, namely, that a simply degenerate hyperbolic manifold homeomorphic to  $S \times [0, \infty)$  is determined by the two following pieces of data:

- 1) The conformal structure on the boundary  $S \times \{0\}$
- 2) the action of the surface group on its limit set.

The action of a simply degenerate Kleinian closed surface group on  $S_\infty^2$  splits naturally into two pieces:

- 1) Action on the domain of discontinuity
- 2) Action on the limit set.

The quotient of the domain of discontinuity by  $\pi_1(S)$  gives rise to a conformal structure on  $S$ . The action on the limit set gives rise (modulo Conjecture 1.1 ) to the ending lamination, described in terms of pre-images of points which have more than one pre-image under the Cannon-Thurston map.

Then by the Ending Lamination Conjecture (and using a result of Sullivan [27]) our conjectural description of hyperbolic structure in terms of asymptotic information follows. The main purpose of this paper is to prove Conjecture 1.1.

In [22] and [25] we had proven the following.

**Theorem 1.2** [22] [25] *Let  $\rho : \pi_1(S) \rightarrow PSL_2(C)$  be a faithful representation of a surface group with or without punctures, and without accidental parabolics. Let  $M = \mathbb{H}^3/\rho(\pi_1(S))$ . Let  $i$  be an embedding of  $S$  in  $M$  that induces a homotopy equivalence. Then the embedding  $\tilde{i} : S \rightarrow \tilde{M} = \mathbb{H}^3$  extends continuously to a map  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ . Further, the limit set of  $\rho(\pi_1(S))$  is locally connected.*

This answers in the affirmative the first part of Conjecture 1.1, generalising work of Cannon-Thurston [7], Minsky [17], McMullen [16], Bowditch [3], Klarreich [13] and the author [21] [23] [24].

The authors of [7] give further an explicit description of this continuous extension in the special case of a fiber of a 3-manifold fibering over the circle. The next theorem is due to Cannon and Thurston [7] (for 3 manifolds fibering over the circle) and Minsky [17] (for bounded geometry closed surface Kleinian groups):

**Theorem 1.3 Cannon-Thurston [7], Minsky [17]** *Suppose a closed surface group  $\pi_1(S)$  of bounded geometry acts freely and properly discontinuously on  $\mathbb{H}^3$  by isometries. Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

In this paper, we generalise the above theorem to arbitrary Kleinian closed surface groups. The main theorem of this paper in combination with Theorem 1.2 is:

**Main Theorem 2.1:** *Suppose a closed surface group  $\pi_1(S)$  acts freely and properly discontinuously on  $\mathbb{H}^3$  by isometries. Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

In passing from Theorem 1.3 to Theorem 2.1 we have just removed the hypothesis of *bounded geometry*. This answers affirmatively the second part of Conjecture 1.1, and resolves one of the issues left unanswered in [22] and [25].

The techniques of this paper can be modified to take care of punctured surface groups, and more generally finitely generated Kleinian groups with core incompressible away from cusps, by using the main theorems of [22] and [25]. However, we restrict ourselves to closed surface groups as this minimises technical complications, describing the necessary generalisations only towards the end.

## 1.2 Outline of the paper

A brief outline of the paper follows.

In Section 2, we prove the easy direction of the Main Theorem, i.e. we show that *end points of leaves of the ending lamination are indeed identified by the Cannon-Thurston Map*. We also describe a toy example - a horodisk - which illustrates some of the features of the hard direction.

We describe in section 3, some basic properties of hyperbolic metric spaces and relative hyperbolicity (Gromov [11], Farb [8]).

In Section 4, we recall some of the basic ideas of [22] and [25], which we shall be needing in the proof of the main theorem. In particular, we recall the construction of the model of *split geometry* from [22] and [25], as well as the construction of a hyperbolic ladder-like set  $B_\lambda$  from [22] that has been a chief player in most of our work on Cannon-Thurston maps [24] [23].

In Section 5, we assemble the proof of the main theorem from these ingredients.

The main ideas of the proof are described in the context of simply degenerate surface groups without accidental parabolics. The modifications

necessary for totally degenerate groups are indicated in the final subsection

## 2 Ideal points are identified by Cannon-Thurston Maps

We would like to know exactly which points are identified by the Cannon-Thurston map, whose existence is assured by Theorem 1.2. Let  $\partial i$  denote restriction to the boundary  $S_\infty^1$  of  $\hat{i}$ , the continuous extension to the boundary in Theorem 1.2.

Reformulating the second part of Conjecture 1.1, we shall prove

**Theorem 2.1** *Let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$  be two distinct points that are identified by the Cannon-Thurston map corresponding to a simply degenerate closed surface group (without accidental parabolics). Then  $a, b$  are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon. Conversely, if  $a, b$  are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon, then  $\partial i(a) = \partial i(b)$ .*

### 2.1 The Easy Direction

It is easy to show the converse direction in Theorem 2.1 above. (cf. Lemma 3.5 of [19]):

**Proposition 2.2** *Let  $u, v$  be either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon. Then  $\partial i(u) = \partial i(v)$ .*

**Proof:** ( cf. Lemma 3.5 of [19]. See also [25] )

Take a sequence of short geodesics  $s_i$  exiting the end. Let  $a_i$  be geodesics in the intrinsic metric on the convex core boundary  $S$  (of  $M$ ) freely homotopic to  $s_i$ . (We use the hypothesis of simple degeneracy very mildly here to fix a base pleated surface.) We assume further that  $a_i$ 's are simple. Join  $a_i$  to  $s_i$  by the shortest geodesic  $t_i$  in  $M$  connecting the two curves. Then the collection  $a_i$  converges to the ending lamination on  $S$ . Also, in the universal cover, we obtain segments  $a_{f_i} \subset \tilde{S}$  which are finite segments whose end-points are identified by the covering map  $P : \tilde{M} \rightarrow M$ . We also assume that  $P$  is injective restricted to the interior of  $a_{f_i}$ 's mapping to  $a_i$ . Similarly

there exist segments  $s_{f_i} \subset \tilde{S}$  which are finite segments whose end-points are identified by the covering map  $P : \tilde{M} \rightarrow M$ . We also assume that  $P$  is injective restricted to the interior of  $a_{f_i}$ 's. The finite segments  $s_{f_i}$  and  $a_{f_i}$  are chosen in such a way that there exist lifts  $t_{1i}, t_{2i}$ , joining end-points of  $a_{f_i}$  to corresponding end-points of  $s_{f_i}$ . The union of these four pieces looks like a trapezium (See figure below, where we omit subscripts for convenience).

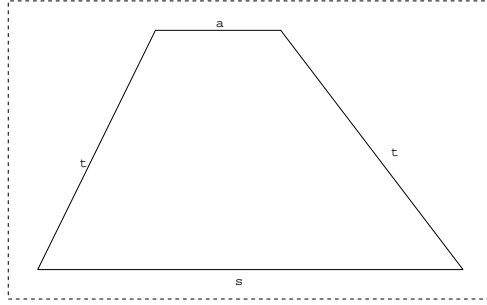


Figure 1: Trapezium

Next, given any leaf  $\lambda$  of the ending lamination, we may choose translates of the finite segments  $a_{f_i}$  (under the action of  $\pi_1(S)$ ) appropriately, such that  $a_{f_i}$  converge to  $\lambda$  in (the Hausdorff/Chabauty topology on closed subsets of)  $\mathbf{H}^2$ . For each  $a_{f_i}$ , let

$$\beta_{f_i} = t_{1i} \circ s_{f_i} \circ \overline{t_{2i}}$$

where  $\overline{t_{2i}}$  denotes  $t_{2i}$  with orientation reversed. Then  $\beta_{f_i}$ 's are uniform hyperbolic quasigeodesics in  $\tilde{M}$ . If the translates of  $a_{f_i}$  we are considering have end-points lying outside large balls around a fixed reference point  $p \in \tilde{S}$ , it is easy to check that  $\beta_{f_i}$ 's lie outside large balls about  $p$  in  $\tilde{M}$ .

At this stage we invoke the existence theorem for Cannon-Thurston maps, Theorem 1.2. Since  $a_{f_i}$ 's converge to  $\lambda$  and there exist uniform hyperbolic quasigeodesics  $\beta_{f_i}$ , joining the end-points of  $a_{f_i}$  and exiting all compact sets, it follows that  $\partial i(u) = \partial i(v)$ , where  $a, b$  denote the boundary points of  $\lambda$ . The Proposition follows from

1. the fact (due to Thurston) that any sequence of simple closed curves on  $S$  with geodesic realisations exiting the end, converges in  $S$  to the ending lamination ([28] Ch. 9)

2. consequently, any leaf of the ending lamination may be realised as the limit of geodesic segments  $a_{f_i}$
3. If we define  $u, v$  to be equivalent if they are the end-points of a leaf of the ending lamination, then the transitive closure of this relation has as elements of an equivalence class
  - a) either ideal end-points of a leaf of a lamination,
  - b) or ideal boundary points of a complementary ideal polygon,
  - c) or a single point in  $S_\infty^1$  which is not an end-point of a leaf of a lamination.

□

**Remark:** Proposition 2.2 shows that the existence of a Cannon-Thurston map automatically guarantees that end-points of leaves of the ending lamination are identified by the Cannon-Thurston map.

### Abstract Ending Laminations

In the next subsection, we explain the ideas that go into a proof of the hard direction of Theorem 2.1, by illustrating them in the special case of bounded geometry. Our approach is different from that of Cannon-Thurston [7] and Minsky [17] inasmuch as we do not use a singular metric on  $\widetilde{M}$ . Instead we use the existence of a quasiconvex hyperbolic ladder  $B_\lambda$ , which has been a chief player in the proofs of the existence of a Cannon-Thurston map in [21], [20], [24], [23], [22]. We have thus, in a sense, reversed the roles of the Ending Lamination and the Cannon-Thurston Map. The approach of Cannon-Thurston starts with a lamination  $\Lambda$ , builds a singular metric on  $\widetilde{M}$ , using  $\Lambda$ , and proves the existence of a Cannon-Thurston map, using this metric (and separating properties of totally geodesic planes in the singular metric.) In our approach, we prove the existence of a Cannon-Thurston map first. This uses the existence of a hyperbolic ladder  $B_\lambda$ . Then, using the existence of a Cannon-Thurston map, we describe point preimages under the  $CT$ -map in terms of the ending lamination  $\Lambda$ . We are thus led naturally to the following.

**Definition 2.3** *Let  $H$  be a finitely presented group acting on a hyperbolic space  $X$  with quotient  $M$ . Let  $X_H$  be a 2-complex with fundamental group  $H$ , and  $i : X_H \rightarrow M$  be a map inducing an isomorphism of fundamental groups. Then  $i$  lifts to  $\tilde{i} : \widetilde{X}_H \rightarrow X$ . A bi-infinite geodesic  $\lambda$  in  $\widetilde{X}_H \subset X$  will be called a **leaf of the abstract ending lamination** for  $i : X_H \rightarrow M$ , if*

1. there exists a set of geodesics  $\sigma_i$  in  $M$  exiting every compact set
2. there exists a set of geodesics  $\alpha_i$  in  $X_H$  with  $i(\alpha_i)$  freely homotopic to  $\sigma_i$
3. there exist finite lifts  $\widetilde{\alpha}_i$  of  $\alpha_i$  in  $(\widetilde{X}_H)$  such that the natural covering map  $\Pi : \widetilde{X}_H \rightarrow X_H$  is injective away from end-points of  $\widetilde{\alpha}_i$
4.  $\widetilde{\alpha}_i$  converges to  $\lambda$  in the Chabauty topology

The theorem that relates leaves of the **abstract ending lamination** to the leaves of the **ending lamination** in the sense of Thurston is given by the following Theorem due to Thurston [28] and Bonahon [2].

**Theorem 2.4 Thurston [28], Bonahon [2]**

*Let  $M$  be a hyperbolic manifold corresponding to a simply degenerate surface group  $\pi_1(S)$  without accidental parabolics. If  $\lambda$  is a leaf of the abstract ending lamination, then its end-points are*

1. either ideal end-points of a leaf of a lamination
2. or ideal boundary points of a complementary ideal polygon

Note that in the definition of an abstract ending lamination, it is not necessary that the  $\sigma_i$ 's be of bounded length, but just that they exit compact sets. The limit of the corresponding  $\alpha_i$ 's is the unique (unmeasured) ending lamination. This is the aspect of the above remarkable theorem that we shall be using.

Also, Proposition 2.2 and its proof readily generalise to

**Proposition 2.5** *Suppose  $H$  is hyperbolic and  $\tilde{i} : \widetilde{X}_H \rightarrow X$  extends to a Cannon-Thurston map on boundaries. Let  $u, v$  be end-points of a leaf of an abstract ending lamination. Then  $\partial i(u) = \partial i(v)$ .*

## 2.2 The Hard Direction: Bounded Geometry

This subsection aims at sketching a proof of the converse to Proposition 2.5 in the special case that  $M$  is a simply degenerate manifold of *bounded geometry* and  $H = \pi_1(S)$ .

Let  $\lambda \subset \tilde{S}$  have ideal points  $\lambda_\infty$  and  $\lambda_{-\infty}$ , which are identified by  $\hat{i}$  (or  $\partial i$ ). We shall show that  $\lambda$  is a leaf of the *abstract ending lamination*.

•1 First, construct the hyperbolic ladder  $B_\lambda$  as in [22]. We briefly recall this construction.  $M$  contains equispaced pleated surfaces  $S_0, S_1, \dots$  exiting the end. Let  $\lambda = \lambda_0 \subset \tilde{S}$ . There exist quasi-isometries  $\phi_i : \tilde{S}_i \rightarrow \tilde{S}_{i+1}$  inducing maps  $\Phi_i$  from geodesics in  $\tilde{S}_i$  to geodesics in  $\tilde{S}_{i+1}$ . Inductively define  $\lambda_{i+1} = \Phi_i(\lambda_i)$  and  $B_\lambda = \bigcup_i \lambda_i$ . Also, let  $\pi_i$  be the nearest-point projection of  $\tilde{S}_i$  onto  $\lambda_i$ . Let  $P$  be obtained by putting the  $\pi_i$ 's together. Then  $P$  turns out to be a large-scale Lipschitz retract of  $\tilde{M}$  onto  $B_\lambda$ , making  $B_\lambda$  a quasiconvex subset. (See [24] for details. Also see Theorem 4.6 below.)

•2 Next, construct a quasigeodesic ray  $r_0$  starting at a fixed base-point  $0 \in \lambda = \lambda_0$  such that  $r_0(i) \in \lambda_i$ . (See Lemma 4.7 below.) Acting on  $r_0$  by various elements  $h$  of  $\pi_1(S)$  we obtain translates of  $r_0$  through any given point  $h \cdot r_0(i) \in B_\lambda, i \in \mathbb{N} \cup \{\infty\}$ . Then  $P(h \cdot r_0)$  lies on  $B_\lambda$  and is a hyperbolic quasigeodesic by Theorem 4.6.

•3 We observe that if  $a_i \rightarrow \lambda_{-\infty}$  and  $b_i \rightarrow \lambda_\infty$ , then geodesics  $\overline{a_i b_i}$  in  $X$ , joining  $a_i, b_i$ , must lie outside large balls about 0 in  $B_\lambda$ .

•4 Next, any  $P(h \cdot r_0)$  is a quasigeodesic ray coarsely separating  $B_\lambda$  into two halves: the left half  $B_\lambda^-$  and  $B_\lambda^+$ . Hence any  $P(\overline{a_i b_i})$  must intersect  $P(h \cdot r_0)$  at some point  $c_i$  provided  $P(h \cdot r_0)(0)$  lies in between  $a_i$  and  $b_i$  on  $\lambda = \lambda_0$ . This shows that any  $P(\overline{a_i b_i})$  converges to the same point on  $\partial X$  as  $a_i, b_i$ . This is because  $c_i$  is joined to  $a_i$  by a part of  $P(\overline{a_i b_i})$  and  $P(\overline{a_i b_i})$  is a hyperbolic quasigeodesic lying outside a large ball about 0. *It is here that we make essential use of the existence of a Cannon-Thurston map, Theorem 1.2.* It follows that

**Any two quasigeodesic rays  $P(h_1 \cdot r_0)$  and  $P(h_2 \cdot r_0)$  are asymptotic.**

Note here that  $P(h_1 \cdot r_0)$  and  $P(h_2 \cdot r_0)$  are projections of translates  $(h_1 \cdot r_0), (h_2 \cdot r_0)$  into  $B_\lambda$  with  $P(h_j \cdot r_0)(i) \in \lambda_i$  for  $j = 1, 2$ .

•5 Further, since quasigeodesics diverge exponentially in a hyperbolic metric space, it follows that there exists  $C_0, A > 0, k > 1$  such that if  $d(P(h_1 \cdot r_0(m)), P(h_2 \cdot r_0(m))) \geq C_0$ , then the length of  $\lambda$  trapped by  $P(h_1 \cdot r_0(0)), P(h_2 \cdot r_0(0))$  has length greater than  $Ak^m$ .

•6 To prove that  $\lambda$  is a **leaf of an abstract ending lamination**, it is

enough that the segments of  $\lambda$  trapped by  $P(h_1 \cdot r_0(0)) (= P(x) = a$  in Figure 2 below) and  $(P(h_2 \cdot r_0(0)) (= P(y) = b$  in Figure 2 below) converge to  $\lambda$ . (As then the geodesics  $[x, y]$  and  $[a, b]$  in  $\tilde{S}$  both converge to  $\lambda$  in the Chabauty topology).

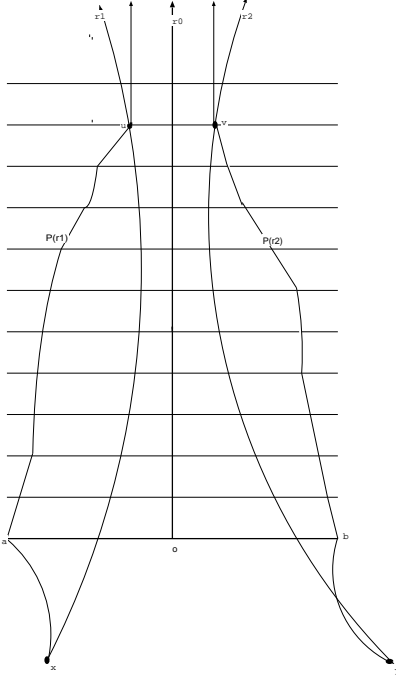


Figure 2: *Point Pre-images are Ending Laminations*

•7 Hence, all that we require are two translates  $(h_1 \cdot r_0), (h_2 \cdot r_0)$  ( $r_1, r_2$  in Figure 2 above) of  $r_0$ , and an arbitrarily large level  $m$  such that  $r_1(m), r_2(m)$  lie on either side of  $r_0(m)$  on  $\lambda_m \subset B_\lambda$ , with  $d(r_i(m), r_j(m)) \geq C_0, i \neq j, i, j \in \{0, 1, 2\}$ . But this can always be ensured by translating  $r_0$  so that

1. setting  $r_i(m) = h_i \cdot r_0(m), i = 1, 2$  under appropriate  $h_i \in \pi_1(S)$
2.  $D_0 \geq d(r_i(m), r_j(m)) \geq C_0, i \neq j, i, j \in \{0, 1, 2\}$  for some fixed (uniform  $D_0 \geq C_0$ ).
3. Defining  $r_i = h_i \cdot r_0$

This completes the sketch of proof in the bounded geometry case. Note that in the above, it is important that the quasigeodesics  $P \cdot r_i$  start diverging

at some large enough level  $m$ . This is ensured by translating  $r_0$  by  $h_i$  and forcing it a certain distance  $\geq C_0$  away from  $r_0$  at height  $m$ , while at the same time requiring that the translate meet  $B_\lambda$  at height  $m$ . Thus we choose  $r_i(m)$  first on  $\lambda_m$  and then take the translate  $r_i$  through  $r_i(m)$ .

**Ingredients of the Proof:**

In brief the main ingredients of the proof are as follows.

1. Existence of a hyperbolic ladder  $B_\lambda = \cup_i \lambda_i$  and retracts  $\Pi_\lambda$  ( $P$  in the sketch above). This was done in [22] for the *split geometry* case, which is what we require in this paper (See Theorem 4.6 below.) An auxiliary metric  $d_G$ , termed the graph-metric is constructed.
2. Existence of a quasigeodesic ray  $r_0 \subset B_\lambda$  such that  $r_0(i) \in \lambda_i$ . This too is ensured by [22] (See Lemma 4.7 below.)
3. for all  $m \in \mathbb{N}$  and  $C_0 \geq 0$ , there exists a constant  $D_0 \geq C_0$  and points  $x_1, x_2$  on either side of  $r_0(m)$  such that

$$C_0 \leq d_G(x_i, r_0(m)) \leq D_0, i = 1, 2$$

. This seemingly simple assertion requires a limiting argument for the graph metric  $d_G$  and is the content of Lemma 5.2 - *CT leaf has infinite diameter*.

4. If  $a_i \rightarrow \lambda_{-\infty}$  and  $b_i \rightarrow \lambda_\infty$  then the geodesic  $\overline{a_i, b_i}$  in  $(\widetilde{M}, d_G)$  lies outside large balls about 0. While this is true in the hyperbolic metric on  $\widetilde{M}$  by Theorem 1.2, it requires a further argument along the lines of (3) above (See Proposition 5.4 below.)
5. Finally, any two quasigeodesics  $r_1, r_2$  lying on  $B_\lambda$  with  $r_i(m) \in \lambda_m$  are asymptotic. (See Proposition 5.8: *Asymptotic Quasigeodesic Rays*.) To show this, we need to show in addition that any such ray  $r_i$  coarsely separates  $B_\lambda$ . We need to show this in the graph-metric  $d_G$ . (See Theorem 5.7 below.)

### 3 Preliminaries and Split Geometry

Let  $(X, d_X)$  be a hyperbolic metric space and  $Y$  be a subspace that is hyperbolic with the inherited path metric  $d_Y$ . By adjoining the Gromov boundaries  $\partial X$  and  $\partial Y$  to  $X$  and  $Y$ , one obtains their compactifications  $\widehat{X}$  and  $\widehat{Y}$  respectively.

Let  $i : Y \rightarrow X$  denote inclusion.

**Definition 3.1** *Let  $X$  and  $Y$  be hyperbolic metric spaces and  $i : Y \rightarrow X$  be an embedding. A **Cannon-Thurston map**  $\hat{i}$  from  $\widehat{Y}$  to  $\widehat{X}$  is a continuous extension of  $i$ .*

#### 3.1 Hyperbolic Metric Spaces

The following Lemma is a ‘quasification’ of the fact that geodesics converging to the same point on  $\partial\Gamma_H$  are asymptotic [See [10], pg. 117].

**Lemma 3.2** *Given  $K \geq 1, \epsilon \geq 0, \delta > 0$  there exists  $\alpha$  such that if  $r_1, r_2 : [0, \infty) \rightarrow X$  are two  $(K, \epsilon)$ -quasigeodesics in a  $\delta$ -hyperbolic metric space  $X$ , converging to the same point in  $\partial X$ , then there exists  $T \geq 0$  such that for all  $t \geq T$*

$$\max \{d(r_1(t), r_2(t)), d(r_1(t), r_2(t))\} \leq \alpha$$

The following is again a simple ‘quasification’ of the standard fact that geodesics diverge exponentially in a  $\delta$ -hyperbolic metric space [1].

**Lemma 3.3** *Given  $K \geq 1, \epsilon \geq 0, D \geq 0$  there exist  $b > 1, A > 0$  and  $C > 0$  such that the following holds:*

*If  $r_1, r_2$  are two  $(K, \epsilon)$ -quasigeodesics with  $d(r_1(0), r_2(0)) \leq D$  and there exists  $T \geq 0$  with  $d(r_1(T), r_2(T)) \geq C$  then any path joining  $r_1(T+t)$  to  $r_2(T+t)$  and lying outside the union of the  $\frac{T+t-1}{K+\epsilon}$ -balls around  $r_1(0), r_2(0)$  has length greater than  $Ab^t$  for all  $t \geq 0$ .*

We shall need to combine Lemmas 3.2 and 3.3 in what follows. Suppose that the  $(K, \epsilon)$  quasigeodesic rays  $r_1, r_2$  converge to the same point  $z$  in  $\partial X$ . Then Lemma 3.2 ensures that they are asymptotic. Parametrizing appropriately, we can assume that  $d(r_1(n), r_2(n)) \leq \alpha$  for all  $n \geq N$ . We now change our vantage point to infinity, the point  $z \in \partial X$  and think of

$r_1, r_2$  as diverging from  $z$ . (This calls for an "inverted parametrization".) Then we shall require a version of Lemma 3.3 that says that if  $r_1, r_2$  start diverging at some  $M \leq N$ , then they diverge exponentially as one goes down to 0. We state this as follows.

**Proposition 3.4** *Given  $K \geq 1, \epsilon \geq 0, \delta \geq 0$ , there exist  $\alpha \geq 0, b > 1, N \in \mathbb{N}, A > 0$  and  $C > 0$  such that the following holds:*

*If  $r_1, r_2$  are two  $(K, \epsilon)$ -quasigeodesics in a  $\delta$ -hyperbolic metric space  $X$ , converging to the same point  $z \in \partial X$ . Then (after reparametrization if necessary)*

1.  $d(r_1(n), r_2(n)) \leq \alpha$  for all  $n \geq N$
2. *If there exists  $M \leq N$  with  $d(r_1(M), r_2(M)) \geq C$  then any path joining  $r_1(0)$  to  $r_2(0)$  has length greater than  $Ab^M$ .*

### 3.2 Relative Hyperbolicity and Electric Geometry

In this subsection, we shall recall first certain notions of relative hyperbolicity due to Farb [8], Klarreich [13] and the author [23].

We consider a hyperbolic metric space  $(X, d)$  and a collection  $\mathcal{H}$  of (*uniformly*)  $C$ -quasiconvex uniformly separated subsets, i.e. there exists  $D > 0$  such that for  $H_1, H_2 \in \mathcal{H}$ ,  $d_X(H_1, H_2) \geq D$ . In this situation  $X$  is hyperbolic relative to the collection  $\mathcal{H}$ . The result in this form is due to Klarreich [13]. We give the general version of Farb's theorem below and refer to [8] and Klarreich [13] for proofs.  $d_{el}$  will denote the **electric metric** (or more precisely the electric pseudometric) on  $X$  obtained by declaring the metric on each  $H \in \mathcal{H}$  to be zero.

**Lemma 3.5** *(See Lemma 4.5 and Proposition 4.6 of [8] and Theorem 5.3 of Klarreich [13]) Given  $\delta, C, D$  there exists  $\Delta$  such that if  $X$  is a  $\delta$ -hyperbolic metric space with a collection  $\mathcal{H}$  of  $C$ -quasiconvex  $D$ -separated sets. then,*

1. *Electric quasi-geodesics electrically track hyperbolic geodesics: Given  $P > 0$ , there exists  $K > 0$  with the following property: Let  $\beta$  be any electric  $P$ -quasigeodesic from  $x$  to  $y$ , and let  $\gamma$  be the hyperbolic geodesic from  $x$  to  $y$ . Then  $\beta \subset N_K^e(\gamma)$ . (Here  $N_K^e$  denotes the  $K$ -neighborhood in the electric metric.)*

2.  $\gamma$  lies in a hyperbolic  $K$ -neighborhood of  $N_0(\beta)$ , where  $N_0(\beta)$  denotes the zero neighborhood of  $\beta$  in the electric metric.
3. *Hyperbolicity:*  $X$  is  $\Delta$ -hyperbolic.

Recall a definition from [23]:

**Definition 3.6** *Given a collection  $\mathcal{H}$  of  $C$ -quasiconvex,  $D$ -separated sets and a number  $\epsilon$  we shall say that a geodesic (resp. quasigeodesic)  $\gamma$  is a geodesic (resp. quasigeodesic) **without backtracking** with respect to  $\epsilon$  neighborhoods if  $\gamma$  does not return to  $N_\epsilon(H)$  after leaving it, for any  $H \in \mathcal{H}$ . A geodesic (resp. quasigeodesic)  $\gamma$  is a geodesic (resp. quasigeodesic) **without backtracking** if it is a geodesic (resp. quasigeodesic) without backtracking with respect to  $\epsilon$  neighborhoods for some  $\epsilon \geq 0$ .*

**Note:** For strictly convex sets,  $\epsilon = 0$  suffices, whereas for convex sets any  $\epsilon > 0$  is enough.

Let  $X$  be a  $\delta$ -hyperbolic metric space, and  $\mathcal{H}$  a family of  $C$ -quasiconvex,  $D$ -separated, collection of subsets. Then by Lemma 3.5,  $X_{el}$  obtained by electrocuting the subsets in  $\mathcal{H}$  is a  $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. Now, let  $\alpha = [a, b]$  be a hyperbolic geodesic in  $X$  and  $\beta$  be an electric  $P$ -quasigeodesic without backtracking joining  $a, b$ . Replace each maximal subsegment, (with end-points  $p, q$ , say) starting from the left of  $\beta$  lying within some  $H \in \mathcal{H}$  by a hyperbolic geodesic  $[p, q]$ . The resulting **connected** path  $\beta_q$  is called an *electro-ambient representative* in  $X$ .

In [23] we noted that  $\beta_q$  need not be a hyperbolic quasigeodesic. However, we did adapt Proposition 4.3 of Klarreich [13] to obtain the following:

**Lemma 3.7** *(See Proposition 4.3 of [13], also see Lemma 3.10 of [23]) Given  $\delta, C, D, P$  there exists  $C_3$  such that the following holds: Let  $(X, d)$  be a  $\delta$ -hyperbolic metric space and  $\mathcal{H}$  a family of  $C$ -quasiconvex,  $D$ -separated collection of quasiconvex subsets. Let  $(X, d_e)$  denote the electric space obtained by electrocuting elements of  $\mathcal{H}$ . Then, if  $\alpha, \beta_q$  denote respectively a hyperbolic geodesic and an electro-ambient  $P$ -quasigeodesic with the same end-points, then  $\alpha$  lies in a (hyperbolic)  $C_3$  neighborhood of  $\beta_q$ .*

## 4 Split Geometry and Hyperbolic Ladders

In this Section, we shall recall the construction of models of *split geometry* from [22]. We shall also need the construction of certain quasiconvex ladder-like sets  $B_\lambda$ .

### 4.1 Split Geometry

Topologically, a **split subsurface**  $S^s$  of a surface  $S$  is a (possibly disconnected, proper) subsurface with boundary such that  $S - S^s$  consists of a non-empty family of non-homotopic annuli, which in turn are not homotopic into the boundary of  $S^s$ .

Geometrically, we assume that  $S$  is given some finite volume hyperbolic structure. A split subsurface  $S^s$  of  $S$  has bounded geometry, i.e.

1. each boundary component of  $S^s$  is of length  $\epsilon_0$ , and is in fact a component of the boundary of  $N_k(\gamma)$ , where  $\gamma$  is a hyperbolic geodesic on  $S$ , and  $N_k(\gamma)$  denotes its  $k$ -neighborhood.
2. For any closed geodesic  $\beta$  on  $S$ , either  $\beta \subset S - S^s$ , or, the length of any component of  $\beta \cap (S - S^s)$  is greater than  $\epsilon_0$ .

Topologically, a **split block**  $B^s \subset B = S \times I$  is a topological product  $S^s \times I$  for some *connected*  $S^s$ . However, its upper and lower boundaries need not be  $S^s \times 1$  and  $S^s \times 0$ . We only require that the upper and lower boundaries be split subsurfaces of  $S^s$ . This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. See figure below:

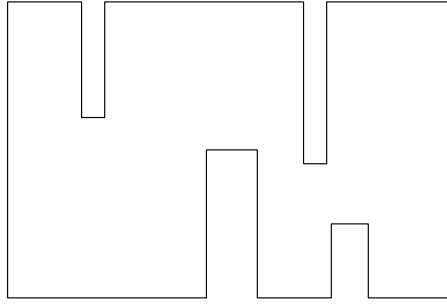


Figure 3: *Split Block with hanging tubes*

Geometrically, we require that the metric on a split block induces a path metric on its *upper and lower horizontal* boundary components, which are subsurfaces of  $S^s \times \partial I$ , such that each horizontal boundary component is a (geometric) split surface. Further, the metric on  $B^s$  induces on each *vertical* boundary component of a Margulis tube  $\partial S^s \times I$  the product metric. Each boundary component for Margulis tubes that ‘travel all the way from the lower to the upper boundary’ is an annulus of height equal to length of  $I$ . We demand further that *hanging tubes* have *length uniformly bounded below by*  $\eta_0 > 0$ . Further, each such annulus has cross section a round circle of length  $\epsilon_0$ . This leaves us to decide the metric on lower and upper boundaries of hanging tubes. Such boundaries are declared to have a metric equal to that on  $S^1 \times [-\eta, \eta]$ , where  $S^1$  is a round circle of length  $\epsilon_0$  and  $\eta$  is a sufficiently small number.

**Note:** In the above definition, we *do not* require that the upper (or lower) horizontal boundary of a split block  $B^s$  be connected for a connected  $B^s$ . This happens due to the presence of *hanging tubes*.

We further require that the distance between horizontal boundary components is at least 1, i.e. for a component  $R$  of  $S^s$   $d(R \times 0, R \times 1) \geq 1$ . We define the **thickness** of a split block to be the supremum of the lengths of  $x \times I$  for  $x \in S^s$  and demand that it be finite (which holds under all reasonable conditions, e.g. a smooth metric; however, since we shall have occasion to deal with possibly discontinuous pseudometrics, we make this explicit). We shall denote the thickness of a split block  $B^s$  by  $l_B$ .

Each component of a split block shall be called a **split component**. We further require that the ‘vertical boundaries’ (corresponding to Euclidean annuli) of split components be uniformly (independent of choice of a block and a split component) quasiconvex in the corresponding split component.

Note that the boundary of each split block has an intrinsic metric that is flat and corresponds to a Euclidean torus.

A lift of a split block to the universal cover of the block  $B = S \times I$  shall be termed a **split component** of  $\tilde{B}$ .

**Remark:** The notion of *split components* we deal with here and in [22] is closely related to the notion of **bands** described by Bowditch in [4], [5] and also to the notion of **scaffolds** introduced by Brock, Canary and Minsky in [6].

We define a **welded split block** to be a split block with identifications as follows: Components of  $\partial S^s \times 0$  are glued together if and only if they correspond to the same geodesic in  $S - S^s$ . The same is done for components of  $\partial S^s \times 1$ . A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also weld the boundary circles of their *lower or upper boundaries* by simply collapsing  $S^1 \times [-\eta, \eta]$  to  $S^1 \times \{0\}$ .

This may be done topologically or geometrically while retaining Dehn twist information about the curves. To record information about the Dehn twists, we have to define (topologically) a map that takes the lower boundary of a welded split block to the upper boundary. We define a map that takes  $x \times 0$  to  $x \times 1$  for every point in  $S^s$ . This clearly induces a map from the lower boundary of a welded split block to its upper boundary. However, this is not enough to give a well-defined map on paths. To do this, we have to record *twist information* about *weld curves*. The way to do this is to define a map on transversals to weld curves. The map is defined on transversals by recording the number of times a transversal to a weld curve  $\gamma \times 0$  twists around  $\gamma \times 1$  on the upper boundary of the welded split block. (A related context in which such transversal information is important is that of markings described in Minsky [18].)

Let the metric product  $S^1 \times [0, 1]$  be called the **standard annulus** if each horizontal  $S^1$  has length  $\epsilon_0$ . For hanging tubes the standard annulus will be taken to be  $S^1 \times [0, 1/2]$ .

Next, we require another pseudometric on  $B$  which we shall term the **tube-electrocuted metric**. We first define a map from each boundary annulus  $S^1 \times I$  (or  $S^1 \times [0, 1/2]$  for hanging annulii) to the corresponding standard annulus that is affine on the second factor and an isometry on the first. Now glue the mapping cylinder of this map to the boundary component. The resulting ‘split block’ has a number of standard annulii as its boundary components. Call the split block  $B^s$  with the above mapping

cylinders attached, the *stabilized split block*  $B^{st}$ .

Glue boundary components of  $B^{st}$  corresponding to the same geodesic together to get the **tube electrocuted metric** on  $B$  as follows. Suppose that two boundary components of  $B^{st}$  correspond to the same geodesic  $\gamma$ . In this case, these boundary components are both of the form  $S^1 \times I$  or  $S^1 \times [0, \frac{1}{2}]$  where there is a projection onto the horizontal  $S^1$  factor corresponding to  $\gamma$ . Let  $S_l^1 \times J$  and  $S_r^1 \times J$  denote these two boundary components (where  $J$  denotes  $I$  or  $[0, \frac{1}{2}]$ ). Then each  $S^1 \times \{x\}$  has length  $\epsilon_0$ . Glue  $S_l^1 \times J$  to  $S_r^1 \times J$  by the natural ‘identity map’. Finally, on each resulting  $S^1 \times \{x\}$  put the zero metric. Thus the annulus  $S^1 \times J$  obtained via this identification has the zero metric in the *horizontal direction*  $S^1 \times \{x\}$  and the Euclidean metric in the *vertical direction*  $J$ . The resulting block will be called the **tube-electrocuted block**  $B_{tel}$  and the pseudometric on it will be denoted as  $d_{tel}$ . Note that  $B_{tel}$  is homeomorphic to  $S \times I$ . The operation of obtaining a *tube electrocuted block and metric*  $(B_{tel}, d_{tel})$  from a split block  $B^s$  shall be called *tube electrocution*.

Next, fix a hyperbolic structure on a Riemann surface  $S$  and construct the metric product  $S \times \mathbb{R}$ . Fix further a positive real number  $l_0$ .

**Definition 4.1** *An annulus  $A$  will be said to be **vertical** if it is of the form  $\sigma \times J$  for  $\sigma$  a geodesic of length less than  $l_0$  on  $S$  and  $J = [a, b]$  a closed sub-interval of  $\mathbb{R}$ .  $J$  will be called the **vertical interval** for the vertical annulus  $A$ .*

*A disjoint collection of annulii is said to be a **vertical system** of annulii if each annulus in the collection is vertical.*

The above definition is based on a definition due to Bowditch [4],[5].

Suppose now that  $S \times \mathbb{R}$  is equipped with a vertical system  $\mathcal{A}$  of annulii. We shall call  $z \in \mathbb{R}$  a

1. a **beginning level** if  $z$  is the lower bound of a vertical interval for some annulus  $A \in \mathcal{A}$ .
2. an **ending level** if  $z$  is the upper bound of a vertical interval for some annulus  $A \in \mathcal{A}$ .

3. an **intermediate level** if  $z$  is an interior point of a vertical interval for some annulus  $A \in \mathcal{A}$ .

In the figure below (where for convenience, all appropriate levels are marked with integers), 2, 5, 11 and 14 are *beginning levels*, 4, 7, 13 and 16 are *ending levels*, 3, 6, 9, 12 and 15 are *intermediate levels*. We shall also allow Dehn twists to occur while going along the annulus.

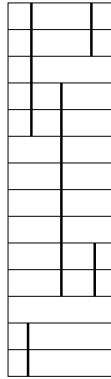


Figure 4: Vertical Annulus Structure

A slight modification of the vertical annulus structure will sometimes be useful.

Replacing each geodesic  $\gamma$  on  $S$  by a neighborhood  $N_\epsilon(\gamma)$  for sufficiently small  $\epsilon$ , we obtain a **vertical Margulis tube** structure after taking products with vertical intervals. The family of Margulis tubes shall be denoted by  $\mathcal{T}$  and the union of their interiors as  $Int\mathcal{T}$ . The union of  $Int\mathcal{T}$  and its horizontal boundaries (corresponding to neighborhoods of geodesics  $\gamma \subset S$ ) shall be denoted as  $Int^+\mathcal{T}$ .

### Thick Block

Fix constants  $D, \epsilon$  and let  $\mu = [p, q]$  be an  $\epsilon$ -thick Teichmüller geodesic of length less than  $D$ .  $\mu$  is  $\epsilon$ -thick means that for any  $x \in \mu$  and any closed geodesic  $\eta$  in the hyperbolic surface  $S_x$  over  $x$ , the length of  $\eta$  is greater than  $\epsilon$ . Now let  $B$  denote the universal curve over  $\mu$  reparametrized such that the length of  $\mu$  is covered in unit time. Thus  $B = S \times [0, 1]$  topologically.

$B$  is given the path metric and is called a **thick building block**.

Note that after acting by an element of the mapping class group, we might as well assume that  $\mu$  lies in some given compact region of Teichmüller space. This is because the marking on  $S \times \{0\}$  is not important, but rather its position relative to  $S \times \{1\}$ . Further, since we shall be constructing models only upto quasi-isometry, we might as well assume that  $S \times \{0\}$  and  $S \times \{1\}$  lie in the orbit under the mapping class group of some fixed base surface. Hence  $\mu$  can be further simplified to be a Teichmüller geodesic joining a pair  $(p, q)$  amongst a finite set of points in the orbit of a fixed hyperbolic surface  $S$ .

### Weak Split Geometry

A manifold  $S \times \mathbb{R}$  equipped with a vertical Margulis tube structure is said to be a model of **weak split geometry**, if it is equipped with a new metric satisfying the following conditions:

1.  $S \times [m, m+1] \cap \text{Int}\mathcal{T} = \emptyset$  (for  $m \in \mathbb{Z} \subset \mathbb{R}$ ) implies that  $S \times [m, m+1]$  is a thick block.
2.  $S \times [m, m+1] \cap \text{Int}\mathcal{T} \neq \emptyset$  (for  $m \in \mathbb{Z} \subset \mathbb{R}$ ) implies that  $S \times [m, m+1] - \text{Int}^+\mathcal{T}$  is (geometrically) a split block.
3. There exists a uniform upper bound on the lengths of vertical intervals for vertical Margulis tubes
4. The metric on each component Margulis tube  $T$  of  $\mathcal{T}$  is hyperbolic

**Note 1:** Dehn twist information can still be implicitly recorded in a model of *weak split geometry* by the Dehn filling information corresponding to tubes  $T$ .

**Note 2:** The metric on a model of *weak split geometry* is possibly discontinuous along the boundary torii of Margulis tubes. If necessary, one could smooth this out. But we would like to carry on with the above metric.

Removing the interiors of Margulis tubes and tube electrocuting each block, we obtain a new pseudo-metric on  $M$  called the **tube electrocuted metric**  $d_{tel}$  on  $M$ . The pseudometric  $d_{tel}$  may also be lifted to  $\tilde{M}$ .

The induced pseudometric on  $\tilde{S}_i$ 's shall be referred to as **split electric metrics**. The notions of *electro-ambient metrics, geodesics and quasi-geodesics* from [22] go through in this context.

Next, we shall describe a **graph metric** on  $\tilde{M}$  which is almost (but not quite) the metric on the nerve of the covering of  $\tilde{M}$  by split components (where each edge is assigned length 1). This is not strictly true as thick blocks are retained with their usual geometry in the graph metric. However the analogy with the nerve is exact if all blocks have *weak split geometry*.

For each split component  $\tilde{K}$  assign a single vertex  $v_K$  and construct a cone of height  $1/2$  with base  $\tilde{K}$  and vertex  $v_K$ . The metric on the resulting space (coned-off or electric space in the sense of Farb [8]) shall be called the **graph metric** on  $\tilde{M}$ .

The union of a split component of  $\tilde{B}$  and the lifts of Margulis tubes (to  $\tilde{M}$ ) that intersect its boundary shall be called a **split amalgamation component** in  $\tilde{M}$ .

**Definition:** A split amalgamation component  $K$  is said to be **( $m, \kappa$ ) - graph quasiconvex** if there exists a  $\kappa$ -quasiconvex (in the hyperbolic metric) subset  $CH(K)$  containing  $K$  such that

1.  $CH(K) \subset N_m^G(K)$  where  $N_m^G(K)$  denotes the  $m$  neighborhood of  $K$  in the graph metric on  $M$ .
2. For each  $K$  there exists  $C_K$  such that  $K$  is  $C_K$ -quasiconvex in  $CH(K)$ .

**Definition:** A model manifold  $M$  of weak split geometry is said to be a model of **split geometry** if there exist  $m, \kappa$  such that each split amalgamation component is  $(m, \kappa)$  - graph quasiconvex.

**Definition:** A block corresponding to  $S \times [m, m+1]$  will be said to start at level  $m$  and end at level  $m+1$ . The corresponding split block  $B^s$  and split components will be said to start and end accordingly. A Margulis tube  $T$  will be said to start at level  $k$  and end at a level  $l > k$  if  $T \cap S_i \neq \emptyset$  if and only if  $k \leq i \leq l$ . If  $k < i < l$ , then  $T$  will be said to intersect the level  $i$ . If  $k \leq i \leq l$ ,  $T$  is said to split the block  $B_i$  and the split surface  $S_i^s$ .

We now state the main theorem of [25].

**Theorem 4.2** [22] [25] *Let  $M^h$  be a 3 manifold homeomorphic to  $S^h \times J$  for  $J = [0, \infty)$  or  $(-\infty, \infty)$  and for  $S^h$  a finite volume hyperbolic surface. Then  $M^h$  has split geometry, where  $S_0^h \subset B_0$  is the lower horizontal surface of the building block  $B_0$ .*

To prove Theorem 4.2 we constructed in [25] the following from the Minsky model for a simply or totally degenerate surface group:

1. A sequence of split surfaces  $S_i^s$  exiting the end(s) of  $M$ . These determine the levels for the split blocks and split geometry. There is a lower bound on the distance between  $S_i^s$  and  $S_{i+1}^s$
2. A collection of Margulis tubes  $\mathcal{T}$ .
3. For each complementary annulus of  $S_i^s$  with core  $\sigma$ , there is a Margulis tube  $T$  whose core is freely homotopic to  $\sigma$  and such that  $T$  intersects the level  $i$ . (What this roughly means is that there is a  $T$  that contains the complementary annulus.)
4. For all  $i$ , either there exists a Margulis tube splitting both  $S_i^s$  and  $S_{i+1}^s$  and hence  $B_i^s$ , or else  $B_i$  is a thick block.
5.  $T \cap S_i^s$  is either empty or consists of a pair of boundary components of  $S_i^s$  that are parallel in  $S_i$ .
6. There is a uniform upper bound  $n$  on the number of surfaces that  $T$  splits.

This ensures that the Minsky model gives rise to a model of *weak split geometry*. We further show that the model constructed is in fact of *split geometry*. To prove this, we showed in addition:

**Proposition 4.3** *Each split component is*

1. *(not necessarily uniformly) quasiconvex in the hyperbolic metric*
2. *uniformly graph-quasiconvex*

Also, let  $d_G$  be the graph metric obtained by electrocuting the collection  $\widetilde{K}$ . Since the components  $\widetilde{K}$  are uniformly graph-quasiconvex by Proposition 4.3 (i.e. each  $CH(\widetilde{K})$  has uniformly bounded diameter in the metric  $d_G$ ), it follows that the identity map from  $(\widetilde{M}, d_2)$  to  $(\widetilde{M}, d_G)$  is a quasi-isometry. Hence, we conclude:

**Proposition 4.4**  $(\widetilde{M}, d_G)$  is a hyperbolic metric space.

## 4.2 Hyperbolic Ladders and Retractions

### Admissible Paths

Admissible paths consist of the following:

1. Horizontal segments along some  $\widetilde{S} \times \{i\}$  for  $i = \{0, 1, 2, 3\}$  (amalgamated blocks) or  $i = \{0, 1\}$  (thick blocks).
2. Vertical segments  $x \times [0, 1]$  or  $x \times [2, 3]$  for amalgamated blocks or  $x \times [0, 1]$  for thick blocks.
3. Vertical segments of length  $\leq l_B$  joining  $x \times \{1\}$  to  $x \times \{2\}$  for amalgamated blocks.

### Ladders in Building Blocks

We shall need to construct a set  $B_\lambda$  containing  $\lambda$  and a retraction  $\Pi_\lambda$  of  $\widetilde{M}$  onto it.  $\Pi_\lambda$  will have the property that it does not stretch distances much. This will show that  $B_\lambda$  is quasi-isometrically embedded in  $\widetilde{M}$ .

To begin, we describe the construction of  $B_\lambda$  restricted to a building block  $B$ .

### Construction of $B_\lambda(B)$ - Thick Block

Let the thick block be the universal curve over a Teichmuller geodesic  $[\alpha, \beta]$ . Let  $S_\alpha$  denote the hyperbolic surface over  $\alpha$  and  $S_\beta$  denote the hyperbolic surface over  $\beta$ .

First, let  $\lambda = [a, b]$  be a geodesic segment in  $\widetilde{S}$ . Let  $\lambda_{B0}$  denote  $\lambda \times \{0\}$ .

Next, let  $\psi$  be the lift of the 'identity' map from  $\widetilde{S}_\alpha$  to  $\widetilde{S}_\beta$ . Let  $\Psi$  denote the induced map on geodesics and let  $\Psi(\lambda)$  denote the hyperbolic geodesic joining  $\psi(a), \psi(b)$ . Let  $\lambda_{B1}$  denote  $\Psi(\lambda) \times \{1\}$ .

For the universal cover  $\widetilde{B}$  of the thick block  $B$ , define:

$$B_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}$$

**Definition:** Each  $\tilde{S} \times i$  for  $i = 0, 1$  will be called a **horizontal sheet** of  $\tilde{B}$  when  $B$  is a thick block.

**Construction of  $B_\lambda(B)$  - Split Block**

First, recall that  $\lambda = [a, b]$  is a geodesic segment in  $\tilde{S}$ . Let  $\lambda_{B0}$  denote  $\lambda \times \{0\}$ .

Next, let  $\lambda_{Gel}$  denote the electric geodesic joining  $a, b$  in the electric pseudo-metric on  $\tilde{S}$  obtained by electrocuting lifts of split components. Let  $\lambda_{B1}$  denote  $\lambda_{Gel} \times \{1\}$ .

Third, recall that  $\tilde{\phi}$  is the lift of a component preserving diffeomorphism  $\phi$  to  $\tilde{S}$  equipped with the electric metric  $d_{Gel}$ . Let  $\tilde{\Phi}$  denote the induced map on geodesics, i.e. if  $\mu = [x, y] \subset (\tilde{S}, d_{Gel})$ , then  $\tilde{\Phi}(\mu) = [\phi(x), \phi(y)]$  is the geodesic joining  $\phi(x), \phi(y)$ . Let  $\lambda_{B2}$  denote  $\tilde{\Phi}(\lambda_{Gel}) \times \{2\}$ .

Fourthly, let  $\Phi(\lambda)$  denote the hyperbolic geodesic joining  $\phi(a), \phi(b)$ . Let  $\lambda_{B3}$  denote  $\Phi(\lambda) \times \{3\}$ .

For the universal cover  $\tilde{B}$  of the thin block  $B$ , define:

$$B_\lambda(B) = \bigcup_{i=0, \dots, 3} \lambda_{Bi}$$

**Definition:** Each  $\tilde{S} \times i$  for  $i = 0 \dots 3$  will be called a **horizontal sheet** of  $\tilde{B}$  when  $B$  is a thick block.

**Modifications for Bi-infinite Geodesics**

Only a few minor modifications are necessary for the above construction to go through for bi-infinite geodesics  $\lambda$ . In both a thick block and a split block, the maps  $\psi$  and  $\phi$  are quasi-isometries (not necessarily uniform for split blocks). This induces maps  $\Psi$  and  $\Phi$  on geodesics. If  $\lambda$  is an infinite geodesic (semi-infinite or bi-infinite), its image is a quasigeodesic and hence lies in a bounded neighborhood of a geodesic, which we denote as  $\Psi(\lambda)$  or  $\Phi(\lambda)$ . (This is well-defined upto asymptote classes, i.e. one geodesic from an asymptote class is chosen.) With this understanding, the construction of  $B_\lambda(B)$  goes through *mutatis mutandis*.

**Construction of  $\Pi_{\lambda, B}$  - Thick Block**

On  $\tilde{S} \times \{0\}$ , let  $\Pi_{B0}$  denote nearest point projection onto  $\lambda_{B0}$  in the path metric on  $\tilde{S} \times \{0\}$ .

On  $\tilde{S} \times \{1\}$ , let  $\Pi_{B1}$  denote nearest point projection onto  $\lambda_{B1}$  in the path metric on  $\tilde{S} \times \{1\}$ .

For the universal cover  $\tilde{B}$  of the thick block  $B$ , define:

$$\Pi_{\lambda, B}(x) = \Pi_{Bi}(x), x \in \tilde{S} \times \{i\}, i = 0, 1$$

### Construction of $\Pi_{\lambda,B}$ - Split Block

**Definition:** Let  $y \in Y$  and let  $\mu_q$  be an electro-ambient representative of an electric geodesic  $\mu_{Gel}$  in  $(Y, d_{Gel})$ . Then  $\pi_e(y) = z \in \mu_q$  if the ordered pair  $\{d_{Gel}(y, \pi_e(y)), d(y, \pi_e(y))\}$  is minimised at  $z$ .

On  $\tilde{S} \times \{0\}$ , let  $\Pi_{B0}$  denote nearest point projection onto  $\lambda_{B0}$ . Here the nearest point projection is taken in the path metric on  $\tilde{S} \times \{0\}$  which is a hyperbolic metric space.

On  $\tilde{S} \times \{1\}$ , let  $\Pi_{B1}$  denote the nearest point projection onto  $\lambda_{B1}$ . Here the nearest point projection is taken in the sense of the definition above, i.e. minimising the ordered pair  $(d_{Gel}, d_{hyp})$  (where  $d_{Gel}, d_{hyp}$  refer to electric and hyperbolic metrics respectively.)

On  $\tilde{S} \times \{2\}$ , let  $\Pi_{B2}$  denote the nearest point projection onto  $\lambda_{B2}$ . Here, again the nearest point projection is taken in the sense of the preceding definition.

Again, on  $\tilde{S} \times \{3\}$ , let  $\Pi_{B3}$  denote nearest point projection onto  $\lambda_{B3}$ . Here the nearest point projection is taken in the path metric on  $\tilde{S} \times \{3\}$  which is a hyperbolic metric space.

For the universal cover  $\tilde{B}$  of the thin block  $B$ , define:

$$\Pi_{\lambda,B}(x) = \Pi_{B_i}(x), x \in \tilde{S} \times \{i\}, i = 0, \dots, 3$$

$\Pi_{\lambda,B}$  is a retract

**Lemma 4.5** (*[22] [25]*) *There exists  $C > 0$  such that the following holds: Let  $x, y \in \tilde{S} \times \{0, 1\} \subset \tilde{B}$  for some block  $B$ . Then  $d(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq Cd(x, y)$ .*

### Construction of $B_\lambda$ and $\Pi_\lambda$

Given a manifold  $M$  of split geometry, we know that  $M$  is homeomorphic to  $S \times J$  for  $J = [0, \infty)$ . There exists a sequence  $I_i$  of intervals and blocks  $B_i$  where the metric on  $S \times I_i$  coincides with that on some building block  $B_i$ . Denote:

- $B_\mu(B_i) = B_{i\mu}$
- $\Pi_{\mu,B_i} = \Pi_{i\mu}$

Now for a block  $B = S \times I$  (thick or split), a natural map  $\Phi_B$  may be defined taking  $\mu = B_\mu(B) \cap \tilde{S} \times \{0\}$  to a geodesic  $B_\mu(B) \cap \tilde{S} \times \{k\} = \Phi_B(\mu)$  where  $k = 1$  or  $3$  according as  $B$  is thick or split. Let the map  $\Phi_{B_i}$  be denoted as  $\Phi_i$  for  $i \geq 0$ .

We start with a reference block  $B_0$  and a reference geodesic segment  $\lambda = \lambda_0$  on the ‘lower surface’  $\tilde{S} \times \{0\}$ . Now inductively define:

- $\lambda_{i+1} = \Phi_i(\lambda_i)$  for  $i \geq 0$
- $B_{i\lambda} = B_{\lambda_i}(B_i)$
- $\Pi_{i\lambda} = \Pi_{\lambda_i, B_i}$
- $B_\lambda = \bigcup_i B_{i\lambda}$
- $\Pi_\lambda = \bigcup_i \Pi_{i\lambda}$

Recall that each  $\tilde{S} \times i$  for  $i = 0 \cdots m$  is called a **horizontal sheet** of  $\tilde{B}$ , where  $m = 1$  or  $3$  according as  $B$  is thick or amalgamated. We will restrict our attention to the union of the horizontal sheets  $\tilde{M}_H$  of  $\tilde{M}$  with the metric induced from the graph model.

Clearly,  $B_\lambda \subset \tilde{M}_H \subset \tilde{M}$ , and  $\Pi_\lambda$  is defined from  $\tilde{M}_H$  to  $B_\lambda$ . Since  $\tilde{M}_H$  is a ‘coarse net’ in  $\tilde{M}$  (equipped with the *graph model metric*), we will be able to get all the coarse information we need by restricting ourselves to  $\tilde{M}_H$ .

By Lemma 4.5, we obtain the fact that each  $\Pi_{i\lambda}$  is a retract. Hence assembling all these retracts together, we have the following basic theorem from [22]:

**Theorem 4.6** [22] *There exists  $C \geq 0$  such that for any geodesic  $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$ , the retraction  $\Pi_\lambda : \tilde{M}_H \rightarrow B_\lambda$  satisfies:*

$$\text{Then } d_{Gel}(\Pi_{\lambda, B}(x), \Pi_{\lambda, B}(y)) \leq C d_{Gel}(x, y) + C.$$

### 4.3 qi Rays

**Lemma 4.7** *There exists  $C \geq 0$  such that for  $x_i \in \lambda_i$  there exists  $x_{i-1} \in \lambda_{i-1}$  with  $d_G(x_i, x_{i-1}) \leq C$ . Hence, for there exists a  $C$ -quasigeodesic ray  $r$  such that  $r(i) \in \lambda_i \subset B_\lambda$  for  $i \geq 0$ .*

Further, by construction of split blocks,  $d_G(x_i, S_{i-1}) = 1$ .

Hence, given  $p \in \lambda_i$  the sequence of points  $x_n, n \in \mathbb{N} \cup \{\infty\}$  with  $x_i = p$  gives by Lemma 4.7 above, a quasigeodesic in the  $d_G$ -metric. Such quasigeodesics shall be referred to as  *$d_G$ -quasigeodesic rays*.

## 5 Laminations

To distinguish between the ending lamination and bi-infinite geodesics whose end-points are identified by  $\partial i$ , we make the following definition.

**Definition 5.1** A **CT leaf**  $\lambda_{CT}$  is a bi-infinite geodesic whose end-points are identified by  $\partial i$ .

An **EL leaf**  $\lambda_{EL}$  is a bi-infinite geodesic whose end-points are ideal boundary points of either a leaf of the ending lamination, or a complementary ideal polygon.

Then to prove the main theorem 2.1 it suffices to show that

- **A CT leaf is an EL leaf.**

## 5.1 Leaves of Laminations

Our first observation is that any semi-infinite geodesic (in the *hyperbolic metric* on  $\tilde{S}$ ) contained in a *CT leaf* in the base surface  $\tilde{S} = \tilde{S} \times \{0\} \subset \tilde{B} = \tilde{B} \times \{0\}$  has infinite diameter in the *graph metric*  $d_G$  restricted to  $\tilde{B}$ .

### Lemma 5.2 CT leaves have infinite diameter

Let  $\lambda_+ (\subset \lambda \subset \tilde{S} \times \{0\} = \tilde{S})$  be a semi-infinite geodesic (in the hyperbolic metric on  $\tilde{S}$ ) contained in a *CT leaf*  $\lambda$ . Then  $dia_G(\lambda_+)$  is infinite (where  $dia_G$  denotes diameter in the graph metric restricted to  $\tilde{S}$ ).

**Proof:** Suppose not. Let  $\lambda_\infty$  be the ideal end-point of  $\lambda_+$ . Also let the finite end-point of  $\lambda_+$  be  $x$ , so that  $\lambda_+ = [x, \lambda_\infty)$ . Then there exist points  $x_n \in \lambda_+$  tending to infinity, and a number  $K > 0$  such that  $d_G(x, x_n) \leq K$ .

Also, without loss of generality, assume that  $\lambda \subset \tilde{S} \times \{0\} \tilde{B}_0$  for the block  $B_0$ . Hence there exist split componets  $\tilde{B}^i \subset \tilde{B}_0$ , such that  $dia_{hyp}(\lambda \cap \tilde{B}^i) \geq i$  (Here,  $dia_{hyp}$  denotes diameter in the hyperbolic metric on  $\tilde{M}$ ). Acting on  $\tilde{B}$  by elements  $h_i$  of the surface group  $\pi_1(S)$ , we may assume that there exist a sequence of segments  $\lambda^i \subset h_i \cdot \lambda$  such that

- 1  $\lambda^i$  is approximately centered about a fixed origin 0, i.e.  $\lambda^i$  pass uniformly close to 0 and end-points of  $\lambda^i$  are at distance  $\geq i$  from 0
- 2 All the  $\lambda^i$ 's lie entirely within a split component  $\tilde{B}^i$

Since  $\tilde{B}^i$  is quasiconvex, it follows that the  $\lambda^i$ 's are uniform quasigeodesics in  $\tilde{M}$ . Hence, in the limit we obtain a bi-infinite quasigeodesic  $\lambda^\infty$  which is a limit in the Chabauty topology of  $h_i \cdot \lambda$ 's. Since the set of *CT leaves* are closed in the Chabauty topology, it follows that  $\lambda^\infty$  is a *CT leaf*.

But, this is a contradiction, as we have noted already that  $\lambda^\infty$  is a quasigeodesic.  $\square$

In fact, the proof of Lemma 5.2 shows the following somewhat stronger assertion.

**Corollary 5.3** *Given  $k \geq 0$ , there exists  $C \geq 0$  such that if  $B = \cup_{0 \leq i \leq k} B_i$  and  $\lambda \subset \tilde{B}$  is a bi-infinite geodesic in the intrinsic metric on  $\tilde{B}$ , whose end-points are identified by the Cannon-Thurston map, then for any split component  $\tilde{B}^i$ ,  $dia_{hyp}(\lambda \cap \tilde{B}^i) \leq C$*

Using Corollary 5.3, we shall now show:

**Proposition 5.4** *There exists a function  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the following holds:*

*Let  $\lambda$  be a CT leaf. If  $a_i, b_i \in \lambda$  be such that  $d(a_i, 0) \geq N$ ,  $d(b_i, 0) \geq N$ , then  $d_G(\overline{a_i b_i}, 0) \geq M(N)$ , where  $d$  denotes the hyperbolic metric on  $M$ .*

**Proof:** Suppose not. Let  $\lambda_+$  and  $\lambda_-$  denote the ideal end points of  $\lambda$ . Then there exists  $C \geq 0$ ,  $a_i \rightarrow \lambda_-$ ,  $b_i \rightarrow \lambda_+$  such that  $d_G(\overline{a_i b_i}, 0) \leq C$ . That is, there exist  $p_i \in \overline{a_i b_i}$  such that  $d_G(0, p_i) \leq C$ . Due to the existence of a Cannon-Thurston map in the hyperbolic metric (Theorem 1.2), we may assume that  $d(0, p_i) \geq i$  (in the hyperbolic metric). Then the hyperbolic geodesic  $\overline{0, p_i}$  passes through at most  $C$  split blocks for every  $i$ . Let  $B = \cup_{0 \leq i \leq C} B_i$  and  $p_i \rightarrow p_\infty$ . Then  $\overline{0, p_i} \subset \tilde{B}$ . But since  $p_i \in \overline{a_i b_i}$ , then the Cannon-Thurston map identifies  $\lambda_-, \lambda_+, p_\infty$ . See Figure below.

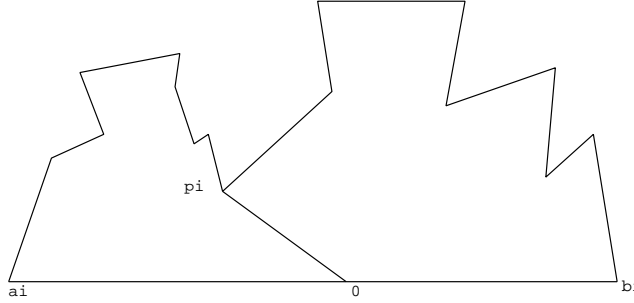


Figure 5: Cannon-Thurston in the Graph Metric

Also,

$$\begin{aligned} \overline{0, p_\infty} &\subset \overline{0, p_\infty} \cup \overline{0, \lambda_+} \\ \overline{0, p_\infty} &\subset \overline{0, p_\infty} \cup \overline{0, \lambda_-} \end{aligned}$$

and at least one of the above two ( $\overline{0, p_\infty} \cup \overline{0, \lambda_+} = \overline{p_\infty, \lambda_+}$  say) must pass close to 0. Then  $\overline{p_\infty, \lambda_+}$  is a CT leaf. But  $\overline{0, p_\infty}$  lies in a  $C$ -neighborhood of 0 in the graph-metric  $d_G$ , contradicting Corollary 5.3 above.  $\square$

## 5.2 Rays Coarsely Separate Ladders

**Definition 5.5** Let  $X, Y, Z$  be geodesically complete metric spaces such that  $X \subset Y \subset Z$ .  $X$  is said to **coarsely separate**  $Y$  into  $Y_1$  and  $Y_2$  if

- (1)  $Y_1 \cup Y_2 = Y$
- (2)  $Y_1 \cap Y_2 = X$
- (3) For all  $M \geq 0$ , there exist  $y_1 \in Y_1$  and  $y_2 \in Y_2$  such that  $d(y_1, Y_2) \geq M$  and  $d(y_2, Y_1) \geq M$
- (4) There exists  $C \geq 0$  such that for all  $y_1 \in Y_1$  and  $y_2 \in Y_2$  any geodesic in  $Z$  joining  $y_1, y_2$  passes through a  $C$ -neighborhood of  $X$ .

See also [9] for a related notion of coarse separation.

We now fix a quasigeodesic ray  $r_0$  as in Lemma 4.7, and consider a translate  $r' = h \cdot r_0$  passing through  $z \in \lambda_m \subset B_\lambda$ , i.e.  $h \cdot r'(m) = z$ . Let  $\Pi_\lambda \cdot r' = r \subset B_\lambda$ . Each  $r(i)$  cuts  $\lambda_i$  into two pieces  $\lambda_i^-$  and  $\lambda_i^+$  with ideal boundary points  $\lambda_{i,-\infty}, \lambda_{i,\infty}$  respectively.

We shall show that  $r$  coarsely separates  $B_\lambda$  into  $B_\lambda^+$  and  $B_\lambda^-$ , where

$$\begin{aligned} B_\lambda^+ &= \bigcup_i \lambda_i^+ \\ B_\lambda^- &= \bigcup_i \lambda_i^- \end{aligned}$$

and  $\lambda_i^+$  (resp.  $\lambda_i^-$ ) is the segment of  $\lambda$  joining  $r(i)$  to the ideal end-point  $\lambda_{i,-\infty}$  (resp.  $\lambda_{i,\infty}$ ).

We need to repeatedly apply Theorem 4.6 to prove the above assertion.

Given  $r'$ , we construct two hyperbolic ladders  $B_\lambda^{+'}$  and  $B_\lambda^{-'}$ , obtained by joining the points  $r'(i)$  to the ideal end-points  $\lambda_{i,-\infty}, \lambda_{i,\infty}$  of  $\lambda_i \subset \tilde{S} \times \{i\}$ . Then  $B_\lambda^{+'}$  and  $B_\lambda^{-'}$  are  $C$ -quasiconvex by Theorem 4.6. Further,  $\Pi_\lambda \cdot r'(i) = r(i)$  by definition of  $r$ . Hence,

$$\begin{aligned} \Pi_\lambda(B_\lambda^{-'}) &= B_\lambda^- \\ \Pi_\lambda(B_\lambda^{+'}) &= B_\lambda^+ \end{aligned}$$

Further,

$$\begin{aligned} B_\lambda^- \cup B_\lambda^+ &= B_\lambda \\ B_\lambda^- \cap B_\lambda^+ &= r \end{aligned}$$

and there exists a  $K_0$  (independent of  $r_0, h, \lambda$ ) such that  $B_\lambda^-, B_\lambda^+, B_\lambda, r$  are all  $K$ -quasiconvex.

Criterion (3) of Definition 5.5 in this context is given by Lemma 5.2 : *CT leaves have infinite diameter.*

To prove that  $r$  separates  $B_\lambda$  into  $B_\lambda^-, B_\lambda^+$ , we need to show first:

**Lemma 5.6** *For all  $K_0 \geq 0$ , there exists  $K_1 \geq 0$  such that if  $p \in B_\lambda^-, q \in B_\lambda^+$  with  $d_G(p, q) \leq K_0$ , then there exists  $z \in r$  such that  $d_G(p, z) \leq K_1$  and  $d_G(q, z) \leq K_1$ .*

**Proof:** Let  $\Pi_\lambda^+$  denote the usual (sheetwise) retract (of Theorem 4.6) onto  $B_\lambda^+$ . Then

$$\Pi_\lambda^+(\lambda_i^-) = r(i)$$

and

$$\Pi_\lambda^+(x) = x$$

for all  $x \in B_\lambda^+$ .

Hence

$$\begin{aligned} \Pi_\lambda^+(q) &= q \\ \Pi_\lambda^+(p) &= z = r(i) \end{aligned}$$

for some  $z \in r$  and some  $i$ .

Therefore, by Theorem 4.6 again,

$$d_G(q, z) \leq C d_g(p, q) = CK_0$$

Choosing  $K_1 = CK_0 + K_0$  (and using the triangle inequality for  $p, q, z$ ) the Lemma follows.  $\square$ .

We are now in a position to prove:

**Theorem 5.7**  *$r$  coarsely separates  $B_\lambda$  into  $B_\lambda^-, B_\lambda^+$ .*

**Proof:** We have already shown

$$\begin{aligned} B_\lambda^- \cup B_\lambda^+ &= B_\lambda \\ B_\lambda^- \cap B_\lambda^+ &= r \end{aligned}$$

and there exists a  $K_0$  (independent of  $r_0, h, \lambda$ ) such that  $B_\lambda^-, B_\lambda^+, B_\lambda, r$  are all  $K$ -quasiconvex.

Criterion (3) of Definition 5.5 is given by Lemma 5.2.

Finally given  $u \in B_\lambda^-$  and  $v \in B_\lambda^+$ , let  $\overline{uv}$  be the geodesic in  $(\widetilde{M}, d_G)$  joining  $u, v$ . Then  $\Pi_\lambda(\overline{uv})$  is a "dotted quasigeodesic"  $u = p_0, p_1, \dots, p_n = v$ , where  $d_G(p_i, p_{i+1}) \leq C$  (and  $C$  is obtained from Theorem 4.6). Further,  $p_0 \in B_\lambda^-, p_n \in B_\lambda^+$  and  $p_i \in B_\lambda$  for all  $i$ . Therefore there exists  $m$  such that  $p_m \in B_\lambda^-, p_{m+1} \in B_\lambda^+$ , with  $d_G(p_m, p_{m+1}) \leq C$ . Hence, by Lemma 5.6, there exists  $K_1 \geq 0$  such that there exists  $z \in r$  with  $d_G(p_m, z) \leq K_1$  and  $d_G(p_{m+1}, z) \leq K_1$ .

Finally, by Proposition 4.4,  $(\widetilde{M}, d_G)$  is hyperbolic, and therefore the "dotted quasigeodesic"  $u = p_0, p_1, \dots, p_n = v$  lies in a uniformly bounded neighborhood of the geodesic  $\overline{uv}$ . That is, there exists  $C_1 \geq 0$  such that for all  $u \in B_\lambda^-$  and  $v \in B_\lambda^+$ , the geodesic  $\overline{uv}$  in  $(\widetilde{M}, d_G)$  joining  $u, v$  passes through a  $C_1$ -neighborhood of  $r$ . This proves (4) in Definition 5.5 and hence we conclude that  $r$  coarsely  $B_\lambda$  into  $B_\lambda^-, B_\lambda^+$ .  $\square$

Next, by Proposition 5.4, we find that if  $a_i, b_i \in \lambda = \lambda_0$  such that  $a_i, b_i$  converge to ideal points  $\lambda_{0,-\infty}, \lambda_{0,\infty}$  (denoted  $\lambda_{-\infty}, \lambda_\infty$  for convenience), then  $\Pi_\lambda(\overline{a_i b_i})$  leaves large balls about 0. Also, by Theorem 5.7 above,  $r$  coarsely separates  $B_\lambda$ . Hence  $\Pi_\lambda(\overline{a_i b_i})$  passes close to  $r(n_i)$  for some  $n_i \in \mathbb{N}$ , where  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Hence, we conclude that any such  $r$  converges on  $\partial\widetilde{M}$  to the same point as  $\partial(\lambda_{-\infty}) = \partial(\lambda_\infty)$ . In particular any two quasigeodesic rays lying on  $B_\lambda$  are asymptotic. We thus have:

### Proposition 5.8 Asymptotic Quasigeodesic Rays

Given  $K \geq 1$  there exists  $\alpha$  such that if  $\lambda$  is a CT-leaf then there exists  $z \in \partial\widetilde{M}$  satisfying the following:

If  $r_1$  and  $r_2$  are  $K$ -quasi-isometric sections contained in  $B_\lambda$  then there exists  $N \in \mathbb{N}$  such that

1.  $r_j(n) \rightarrow z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  as  $n \rightarrow \infty$ , for  $j = 1, 2$ .
2.  $d_G(r_1(n), r_2(n)) \leq \alpha$  for all  $n \geq N$

## 5.3 Main Theorem

We are now in a position to prove the main Theorem of this paper.

**Main Theorem 2.1** *Let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$  be two distinct points that are identified by the Cannon-Thurston map corresponding to a simply degenerate closed surface group (without accidental parabolics). Then  $a, b$  are either ideal end-points of a leaf of the ending lamination (in the sense of Thurston), or ideal boundary points of a complementary ideal polygon. Conversely, if  $a, b$  are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon, then  $\partial i(a) = \partial i(b)$ .*

**Proof of Main Theorem 2.1:** The converse direction has been shown in Proposition 2.2.

Next, let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$ . Then  $(a, b) = \lambda \subset \widetilde{S}_0 \subset \widetilde{M}$  is a CT-leaf.

$(\widetilde{M}, d_G)$  is a hyperbolic metric space by Proposition 4.4. By Lemma 4.7, there exists a quasi-geodesic ray  $r = r_0 \subset B_\lambda$  with  $r(i) \in \lambda_i$ . Then for any two  $\Pi_\lambda(h_i \cdot r) = r_i$ ,  $i = 1, 2$ , Proposition 5.8 shows that there exists  $\alpha, N \in \mathbb{N}$  and  $z \in \partial \widetilde{M}$  satisfying the following:

1.  $r_j(n) \rightarrow z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  as  $n \rightarrow \infty$ , for  $j = 0, 1, 2$ .
2.  $d_G(r_i(n), r_j(n)) \leq \alpha$  for all  $n \geq N$

From Proposition 3.4 there exist  $b > 1$ ,  $A > 0$  and  $C > 0$  such that if  $d_G(r_i(M), r_j(M)) \geq C$  for some  $M \in \mathbb{N}$ , then the part of  $\lambda = \lambda_0$  intercepted between  $r_i(0)$  and  $r_j(0)$  has length greater than  $Ab^M$ .

Next, by Lemma 5.2, it follows that for each  $i$ , if  $\lambda_i^+$  and  $\lambda_i^-$  denote the two pieces of  $\lambda_i \setminus r(i)$ , then there exist  $u_i, v_i \in \lambda_i^-, \lambda_i^+$  such that  $d_G(u_i, r(i)) = d_G(v_i, r(i)) = C$ . Fix such an  $i$  for the time being.

Let  $h_j \in \pi_1(S)$ ,  $j = 1, 2$  denote elements that translate  $r(m)$  to  $u_i, v_i$  respectively. Let

$$\begin{aligned} \Pi_\lambda(h_j \cdot r) &= r_j, j = 1, 2 \\ r_1(0) &= a_i, r_2(0) = b_i \\ h_1 \cdot r(0) &= x_i, h_2 \cdot r(0) = y_i \end{aligned}$$

Then the geodesics  $[a_i, b_i]$  and  $[x_i, y_i]$  converge to  $\lambda$  in the Chabauty topology on (closed subsets of)  $\widetilde{S}$ . Next,  $[x_i, y_i]$  are connected to  $u_i, v_i$  by translates of  $r$ . Now,  $d_G(u_i, v_i) \leq 2C$ . Let  $\overline{u_i, v_i}^{ea}$  denote the electro-ambient quasigeodesic in  $\widetilde{M}$  joining  $u_i, v_i$  (see Lemma 3.7 and discussion preceding).

Then by uniform graph-quasiconvexity of split components (Proposition 4.3), there is a uniform  $L_0$  such that  $\overline{u_i, v_i}^{ea}$  lies outside an  $(i-L_0)$ -neighborhood of  $\tilde{S} \times \{0\}$  in the graph metric. Further by Lemma 3.7 (see also Lemma 2.1 of Bonahon [2]), there exists  $\epsilon > 0$  such that the hyperbolic geodesic  $\overline{u_i, v_i}^{hyp}$  joining  $u_i, v_i$  in  $\tilde{M}$  lies in a (hyperbolic)  $\epsilon$  neighborhood of  $\overline{u_i, v_i}^{ea}$ . In particular,  $\overline{u_i, v_i}^{hyp}$  exits the end as  $i \rightarrow \infty$ . Hence  $[x_i, y_i]$  converge to a leaf of an abstract ending lamination. Therefore  $\lambda$  is a leaf of an abstract ending lamination.

Hence finally by Theorem 2.4,  $a, b$  are either ideal end-points of a leaf of the ending lamination (in the sense of Thurston), or ideal boundary points of a complementary ideal polygon.  $\square$

**Totally Degenerate Groups** We have so far assumed, for ease of exposition, that we are dealing with simply degenerate groups. A few words are in order as to the necessary modifications to be made for the totally degenerate case. All that we need to do to generalise to this situation is fix a base reference surface. This is the only place where we used the boundary of the convex core as a preferred base reference surface, but any fixed pleated surface would do as well. The same works for 3-manifolds with incompressible boundary. Here, we fix a reference base surface for each end, and use the existence of Cannon-Thurston maps, ladders  $B_\lambda$  and rays from [22].

**Surfaces with punctures** This is a bit more involved and requires the use of the **partial electrocution** techniques of [22]. But given this, the same proof goes through *mutatis mutandis* more generally to 3-manifolds whose cores are *incompressible away from cusps*.

**Problem:** Generalise to 3-manifolds with compressible core. A first (and representative) case is for free geometrically infinite groups without parabolics.

## 5.4 Limit Set and Limiting $\mathbb{R}$ -tree

Recall from [22] that each block in the split geometry model of  $\tilde{M}$  corresponds to a splitting of the surface group, and hence an action of  $\pi_1(S)$  on a simplicial tree. As  $i \rightarrow \infty$ , the split blocks  $B_i^s$  and hence the induced splittings also go to infinity, converging to a **free action of the surface group on an  $\mathbb{R}$ -tree  $T_L$  dual to the ending lamination  $\mathcal{L}$** . Let  $\bar{T}_L$  denote the (Gromov) compactification of  $T_L$ . Then Theorem 2.1 shows that the limit set  $\Lambda$  is equivariantly (under the action of  $\pi_1(S)$ ) homeomorphic to  $\bar{T}_L$ .

Further, split blocks contain short curves  $\sigma_i$ , which correspond to actions of  $\pi_1(S)$  on simplicial trees  $T_i$  (arising from splittings of  $\pi_1(S)$ ). Thus, we have shown that any such sequence of actions of  $\pi_1(S)$  on  $T_i$  converges to the action of  $\pi_1(S)$  on  $\Lambda$ .

In [12], Kent, Leininger and Schleimer describe a curve complex version of the Birman Exact Sequence for mapping class groups.

If  $S$  is a closed surface with genus at least two, and  $\dot{S}$  is the surface  $S$  punctured once, then filling in the puncture (generically) induces a map  $P_{CC}$  at the level of the curve complex  $\mathbb{C}$  from  $\mathbb{C}(\dot{S}) \rightarrow \mathbb{C}(S)$ .

It is interesting to note that the trees  $T_i$  almost always (the exceptions being due to the presence of thick blocks) correspond to the fibers of this map over a geodesic ray  $\bar{r}$ , where  $\bar{r} \subset \mathbb{C}(S)$  joins a base-point in  $\mathbb{C}(S)$  to the ending lamination  $\mathcal{L}$ , which in turn is a boundary point of  $\mathbb{C}(S)$ . Hence the manifold  $\widetilde{M}$  equipped with the graph metric  $d_G$  is intimately connected to the preimage of  $\bar{r}$  under  $P_{CC}$ .

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